

Problem 6)

$$e^{i\theta} + e^{2i\theta} + \dots + e^{in\theta} = x(1 + x + x^2 + \dots + x^{n-1}) \quad \leftarrow x = e^{i\theta}$$

$$= x \frac{x^n - 1}{x - 1} \quad \leftarrow \text{geometric series}$$

$$= e^{i\theta} \frac{e^{in\theta} - 1}{e^{i\theta} - 1} = e^{i\theta} \frac{e^{in\theta/2} (e^{in\theta/2} - e^{-in\theta/2})}{e^{i\theta/2} (e^{i\theta/2} - e^{-i\theta/2})}$$

$$= e^{i(n+1)\theta/2} \frac{\sin(n\theta/2)}{\sin(\theta/2)}$$

Therefore,

$$(\cos\theta + \cos 2\theta + \dots + \cos n\theta) + i(\sin\theta + \sin 2\theta + \dots + \sin n\theta) =$$

$$\frac{\sin(n\theta/2)}{\sin(\theta/2)} [\cos(n+1)\theta/2 + i \sin(n+1)\theta/2]$$

$$\Rightarrow \cos\theta + \cos 2\theta + \dots + \cos n\theta = \frac{\sin(n\theta/2) \cos(n+1)\theta/2}{\sin(\theta/2)},$$

$$\sin\theta + \sin 2\theta + \dots + \sin n\theta = \frac{\sin(n\theta/2) \sin(n+1)\theta/2}{\sin(\theta/2)}.$$

$$\text{Consequently: } \frac{\sin\theta + \sin 2\theta + \dots + \sin n\theta}{\cos\theta + \cos 2\theta + \dots + \cos n\theta} = \tan\left[\frac{(n+1)\theta}{2}\right]. \quad \checkmark$$