Problem 5) $\int_a^b [f(x)g(x)]'\mathrm{d}x = \int_a^b f'(x)g(x)\mathrm{d}x + \int_a^b f(x)g'(x)\mathrm{d}x$

$$\rightarrow \quad \int_a^b f'(x)g(x)\mathrm{d}x = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)\mathrm{d}x$$

$$= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\mathrm{d}x.$$

a) $$\int_0^\infty xe^{-\kappa x}\mathrm{d}x = \cancelto{0}{-\frac{x}{\kappa}e^{-\kappa x}\Big|_0^\infty} + \frac{1}{\kappa}\int_0^\infty e^{-\kappa x}\mathrm{d}x = -\frac{1}{\kappa^2}e^{-\kappa x}\Big|_0^\infty = \frac{1}{\kappa^2}.$$

b) $$\int_0^\infty x^2e^{-\kappa x}\mathrm{d}x = \cancelto{0}{-\frac{x^2}{\kappa}e^{-\kappa x}\Big|_0^\infty} + \frac{2}{\kappa}\int_0^\infty xe^{-\kappa x}\mathrm{d}x = \frac{2}{\kappa^3}.$$

c) $$\int_0^\infty \sin(\omega x)\,e^{-\kappa x}\mathrm{d}x = \cancelto{0}{-\frac{\sin(\omega x)e^{-\kappa x}}{\kappa}\Big|_0^\infty} + \frac{\omega}{\kappa}\int_0^\infty \cos(\omega x)\,e^{-\kappa x}\mathrm{d}x$$

$$= \frac{\omega}{\kappa}\left[-\frac{\cos(\omega x)e^{-\kappa x}}{\kappa}\Big|_0^\infty - \frac{\omega}{\kappa}\int_0^\infty \sin(\omega x)\,e^{-\kappa x}\mathrm{d}x\right]$$

$$= \frac{\omega}{\kappa}\left[\frac{1}{\kappa} - \frac{\omega}{\kappa}\int_0^\infty \sin(\omega x)\,e^{-\kappa x}\mathrm{d}x\right]$$

$$\rightarrow \quad \left(1 + \frac{\omega^2}{\kappa^2}\right)\int_0^\infty \sin(\omega x)\,e^{-\kappa x}\mathrm{d}x = \frac{\omega}{\kappa^2} \quad \rightarrow \quad \int_0^\infty \sin(\omega x)\,e^{-\kappa x}\mathrm{d}x = \frac{\omega}{\omega^2+\kappa^2}.$$

d) In part (c), we also found that $\int_0^\infty \cos(\omega x)\,e^{-\kappa x}\mathrm{d}x = (\kappa/\omega)\int_0^\infty \sin(\omega x)\,e^{-\kappa x}\mathrm{d}x$. Therefore,

$$\int_0^\infty \cos(\omega x)\,e^{-\kappa x}\mathrm{d}x = \frac{\kappa}{\omega^2+\kappa^2}.$$

e) $$\int_0^\infty x^2e^{-\pi x^2}\mathrm{d}x = \cancelto{0}{-\frac{x}{2\pi}e^{-\pi x^2}\Big|_0^\infty} + \frac{1}{2\pi}\int_0^\infty e^{-\pi x^2}\mathrm{d}x = \frac{1}{4\pi}. \quad \leftarrow \text{(see Problem 4)}$$

f) $$\int_0^\infty x^3e^{-\pi x^2}\mathrm{d}x = \cancelto{0}{-\frac{x^2}{2\pi}e^{-\pi x^2}\Big|_0^\infty} + \int_0^\infty \frac{2x}{2\pi}e^{-\pi x^2}\mathrm{d}x = -\frac{1}{2\pi^2}e^{-\pi x^2}\Big|_0^\infty = \frac{1}{2\pi^2}.$$

g) $$\int_0^a x\ln(x)\,\mathrm{d}x = \frac{x^2}{2}\ln(x)\Big|_0^a - \int_0^a (x^2/2)(1/x)\mathrm{d}x = \frac{a^2}{2}\ln(a) - \frac{x^2}{4}\Big|_0^a = \frac{a^2}{2}[\ln(a) - \tfrac{1}{2}].$$

h) $$\int_0^a x^\kappa \ln^2(x)\,\mathrm{d}x = \frac{x^{\kappa+1}}{\kappa+1}\ln^2(x)\Big|_0^a - \int_0^a \left(\frac{x^{\kappa+1}}{\kappa+1}\right)\left(\frac{2}{x}\ln x\right)\mathrm{d}x$$

$$= \frac{a^{\kappa+1}\ln^2(a)}{\kappa+1} - \frac{2}{\kappa+1}\int_0^a x^\kappa \ln(x)\,\mathrm{d}x$$

$$= \frac{a^{\kappa+1}\ln^2(a)}{\kappa+1} - \frac{2}{\kappa+1}\left[\frac{x^{\kappa+1}}{\kappa+1}\ln(x)\Big|_0^a - \int_0^a \left(\frac{x^{\kappa+1}}{\kappa+1}\right)\left(\frac{1}{x}\right)\mathrm{d}x\right]$$

$$= \frac{a^{\kappa+1}\ln^2(a)}{\kappa+1} - \frac{2a^{\kappa+1}\ln(a)}{(\kappa+1)^2} + \frac{2a^{\kappa+1}}{(\kappa+1)^3}.$$

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i) The integral is an even function of κ. To see this, make the change of variable $\theta = \pi - \varphi$, then observe that, aside from switching from κ to $-\kappa$, the integral remains the same; that is,

$$\int_0^\pi \frac{\sin^3\theta}{(1-\kappa\cos\theta)^3}\mathrm{d}\theta = -\int_\pi^0 \frac{\sin^3(\pi-\varphi)}{[1-\kappa\cos(\pi-\varphi)]^3}\mathrm{d}\varphi = \int_0^\pi \frac{\sin^3\varphi}{(1+\kappa\cos\varphi)^3}\mathrm{d}\varphi. \tag{1}$$

At $\kappa = 0$, the integral is substantially simplified and may be readily evaluated, as follows:

$$\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \int_0^\pi \sin\theta\,(1-\cos^2\theta)\mathrm{d}\theta = (-\cos\theta + ⅓\cos^3\theta)|_{\theta=0}^{\pi} = 4/3. \tag{2}$$

It is thus necessary only to evaluate the integral for $\kappa > 0$. To this end, we use the method of integration by parts, choosing $f_1'(\theta) = \sin\theta/(1-\kappa\cos\theta)^3$ and $g_1(\theta) = \sin^2\theta$. Considering that $f_1(\theta) = -(1-\kappa\cos\theta)^{-2}/(2\kappa)$ and $g_1'(\theta) = 2\sin\theta\cos\theta$, we will have

$$\int_0^\pi \frac{\sin^3\theta}{(1-\kappa\cos\theta)^3}\mathrm{d}\theta = -\frac{\sin^2\theta}{2\kappa(1-\kappa\cos\theta)^2}\Big|_{\theta=0}^{\pi} + \int_0^\pi \frac{2\sin\theta\cos\theta}{2\kappa(1-\kappa\cos\theta)^2}\mathrm{d}\theta = \frac{1}{\kappa}\int_0^\pi \frac{\sin\theta\cos\theta}{(1-\kappa\cos\theta)^2}\mathrm{d}\theta. \tag{3}$$

To evaluate the remaining integral, let $f_2'(\theta) = \sin\theta/(1-\kappa\cos\theta)^2$ and $g_2(\theta) = \cos\theta$, which yield $f_2(\theta) = -(1-\kappa\cos\theta)^{-1}/\kappa$ and $g_2'(\theta) = -\sin\theta$. Consequently,

$$\int_0^\pi \frac{\sin\theta\cos\theta}{(1-\kappa\cos\theta)^2}\mathrm{d}\theta = -\frac{\cos\theta}{\kappa(1-\kappa\cos\theta)}\Big|_{\theta=0}^{\pi} - \frac{1}{\kappa}\int_0^\pi \frac{\sin\theta}{1-\kappa\cos\theta}\mathrm{d}\theta$$

$$= \frac{1}{\kappa(1+\kappa)} + \frac{1}{\kappa(1-\kappa)} - \frac{1}{\kappa^2}\overbrace{\ln(1-\kappa\cos\theta)}^{\boxed{\text{valid for } |\kappa| \le 1}}\Big|_{\theta=0}^{\pi} = \frac{2}{\kappa(1-\kappa^2)} - \frac{1}{\kappa^2}\ln\left(\frac{1+\kappa}{1-\kappa}\right). \tag{4}$$

Combining Eqs.(3) and (4), we finally obtain

$$\int_0^\pi \frac{\sin^3\theta}{(1-\kappa\cos\theta)^3}\mathrm{d}\theta = \frac{2}{\kappa^2(1-\kappa^2)} - \frac{1}{\kappa^3}\ln\left(\frac{1+\kappa}{1-\kappa}\right); \qquad |\kappa| \le 1. \tag{5}$$

Digression: As a check on the above result, consider the case of $\kappa = 0$, where the integral simplifies to $\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = 4/3$. To confirm that Eq.(5) does in fact yield the correct result in the limit when $\kappa \to 0$, observe that, for sufficiently small κ, one may invoke the geometric series identity $1/(1-\kappa^2) = 1 + \kappa^2 + \kappa^4 + \cdots$ as well as the Taylor series expansion $\ln(1\pm\kappa) = \pm\kappa - ½\kappa^2 \pm ⅓\kappa^3 - ¼\kappa^4 \pm ⅕\kappa^5 + \cdots$. We thus arrive at

$$\frac{2}{\kappa^2(1-\kappa^2)} - \frac{1}{\kappa^3}\ln\left(\frac{1+\kappa}{1-\kappa}\right) = \frac{2(1+\kappa^2+\kappa^4+\cdots)}{\kappa^2} - \frac{1}{\kappa^3}[(\kappa - ½\kappa^2 + ⅓\kappa^3 - ¼\kappa^4 + ⅕\kappa^5 - \cdots)$$

$$-(-\kappa - ½\kappa^2 - ⅓\kappa^3 - ¼\kappa^4 - ⅕\kappa^5 - \cdots)]$$

$$= \cancel{\frac{2}{\kappa^2}} + 2 + 2\kappa^2 + \cdots - \cancel{\frac{2}{\kappa^2}} - ⅔ - ⅖\kappa^2 - \cdots, \qquad |\kappa| < 1. \tag{6}$$

As expected, in the limit of $\kappa \to 0$, the preceding expression approaches $4/3$.

Equation (5) also indicates that the integral is an even function of κ which diverges at $\kappa = \pm 1$. This is consistent with the fact that, at $\kappa = 1$, we have

$$\int_0^\pi \left(\frac{\sin\theta}{1-\cos\theta}\right)^3 \mathrm{d}\theta = \int_0^\pi \left[\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}\right]^3 \mathrm{d}\theta = \int_0^\pi \cot^3(\theta/2)\,\mathrm{d}\theta = 2\int_0^{\pi/2} \overset{\boxed{\text{non-integrable singularity at } \varphi = 0}}{\overset{\downarrow}{\cot^3}} \varphi\,\mathrm{d}\varphi. \tag{7}$$

It remains to evaluate the integral for $\kappa > 1$. Returning to Eq.(4), we now write

$$\begin{aligned}
\int_0^\pi \left(\frac{\sin\theta}{1-\kappa\cos\theta}\right)\mathrm{d}\theta &= \int_0^{\arccos(1/\kappa)} \left(\frac{\sin\theta}{1-\kappa\cos\theta}\right)\mathrm{d}\theta + \int_{\arccos(1/\kappa)}^{\pi} \left(\frac{\sin\theta}{1-\kappa\cos\theta}\right)\mathrm{d}\theta = \\
&= \frac{1}{\kappa}\ln(1-\kappa\cos\theta)\Big|_{\theta=0}^{\arccos(1/\kappa)} + \frac{1}{\kappa}\ln(1-\kappa\cos\theta)\Big|_{\theta=\arccos(1/\kappa)}^{\pi} \\
\boxed{\begin{matrix}\text{Principal value}\\ (0<\varepsilon\ll 1)\end{matrix}} \rightarrow &= \frac{1}{\kappa}[\ln(-\varepsilon) - \ln(1-\kappa)] + \frac{1}{\kappa}[\ln(1+\kappa) - \ln(\varepsilon)] \\
&= \frac{1}{\kappa}[\ln\cancel{(\varepsilon)} + \cancel{i\pi} - \ln(\kappa-1) - \cancel{i\pi}] + \frac{1}{\kappa}[\ln(1+\kappa) - \ln\cancel{(\varepsilon)}] = \frac{1}{\kappa}\ln\left(\frac{\kappa+1}{\kappa-1}\right).
\end{aligned} \tag{8}$$

Substitution into Eq.(4) and combining the result with Eq.(3), one arrives at

$$\int_0^\pi \frac{\sin^3\theta}{(1-\kappa\cos\theta)^3}\mathrm{d}\theta = \frac{2}{\kappa^2(1-\kappa^2)} - \frac{1}{\kappa^3}\ln\left(\frac{\kappa+1}{\kappa-1}\right); \qquad \kappa > 1. \tag{9}$$

As was the case with Eq.(5), the integral is seen to be an even function of κ.
