Problem 4) Let $a = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$. Then

$$a^2 = \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)^2 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx \int_{-\infty}^{\infty} e^{-\pi y^2} dy = \iint_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx\,dy$$

$$= \int_0^{\infty} 2\pi r e^{-\pi r^2} dr = -e^{-\pi r^2}\Big|_0^{\infty} = 1. \checkmark \Rightarrow a = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1. \checkmark$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2 + 2\pi x_0 x} dx = \int_{-\infty}^{\infty} e^{-\pi(x-x_0)^2 + \pi x_0^2} dx = e^{+\pi x_0^2} \int_{-\infty}^{\infty} e^{-\pi(x-x_0)^2} dx$$

$$= e^{\pi x_0^2} \int_{-\infty}^{\infty} e^{-\pi x^2} dx = e^{\pi x_0^2}$$