

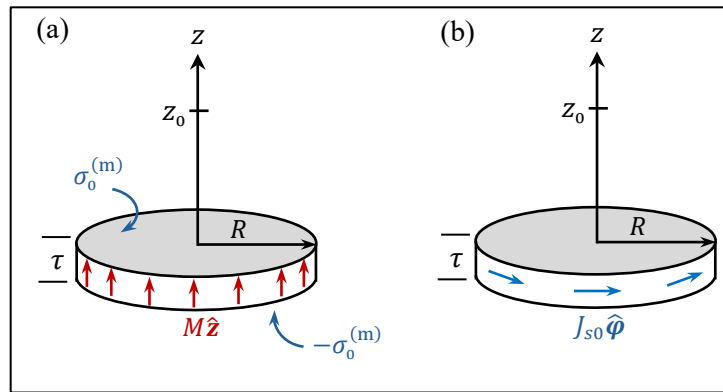
Please write your name and ID number on all the pages, then staple them together.

Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

- 2 pts **Problem 1)** a) Invoke Newton's second law of motion, $\mathbf{f} = m\mathbf{a}$, and the Lorentz force law of classical electrodynamics, $\mathbf{f} = q(\mathbf{E} + \mathbf{V} \times \mathbf{B})$, to relate the *SI* (or *MKSA*) units [volt] and [weber] to those of length [meter], mass [kilogram], time [second], and electric current [ampère].
- 2 pts b) If a magnetic dipole moment is defined as $\mathbf{m} = q^{(m)}\mathbf{d}$, its units will be [weber · meter]. In the alternative (Ampèrian) definition, $\mathbf{m} = \mu_0 I A \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the loop area A , the units of $\mathbf{m}$ are [(henry/meter) · ampère · meter²]. Invoke the defining identity $\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$ to confirm that the two definitions of $\mathbf{m}$ have identical units.
- 2 pts c) The units of capacitance C are [farad] and the units of inductance L are [henry]. Considering the relations $Q = CV$ and $\Phi = LI$, where Q is electric charge, V is voltage, Φ is magnetic flux (i.e., $\int \mathbf{B} \cdot d\mathbf{s}$), and I is electric current, relate [farad] and [henry] to the fundamental *SI* units.

Problem 2) The figure below shows a uniformly magnetized disk of radius R , thickness τ , and magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The goal here is to compute the magnetic field $\mathbf{H}$ on the z -axis at a distance z_0 from the upper surface of the disk. (**Note:** z_0 can be positive or negative.) Recall that the magnetization $\mathbf{M}$ can be treated in terms of the magnetic surface charge densities $\pm\sigma_0^{(m)}$ [weber/m²], as in figure (a), or, alternatively, in terms of an electric surface current density $\mathbf{J}_{s0} = J_{s0}\hat{\boldsymbol{\phi}}$ [ampère/m], as in figure (b).

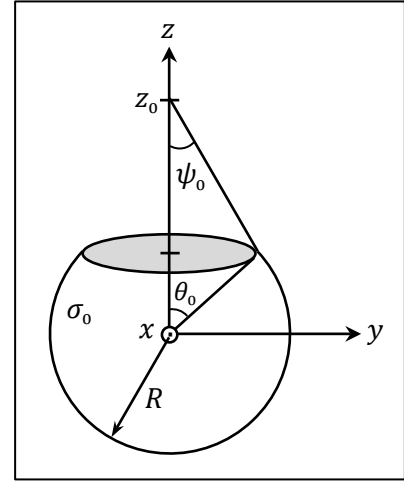


- 1 pt a) Considering that the magnetic dipole moment is defined as $\mathbf{m} = q^{(m)}\mathbf{d}$ for a pair of magnetic point charges $\pm q^{(m)}$ separated by a distance $\mathbf{d}$, show that, in figure (a), $\sigma_0^{(m)} = M$.
- 1 pt b) Considering that the magnetic dipole moment is defined as $\mathbf{m} = \mu_0 I A \hat{\mathbf{n}}$ for a current loop of area A , electric current I , and surface normal $\hat{\mathbf{n}}$, show that, in figure (b), $M = \mu_0 J_{s0}$.
- 3 pts c) Invoke the similarity between Maxwell's 1st and 4th equations, namely, $\epsilon_0 \nabla \cdot \mathbf{E} = \rho_{\text{free}} + \rho_{\text{bound}}^{(e)}$ and $\mu_0 \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} = \rho_{\text{bound}}^{(m)}$ to argue that, in parallel with the Coulomb law for the $\mathbf{E}$ -field of an electric point charge $q^{(e)}$, the magnetic field of the magnetic point charge $q^{(m)}$ at a distance $\mathbf{r}$ is $\mathbf{H}(\mathbf{r}) = q^{(m)}\hat{\mathbf{r}} / (4\pi\mu_0 r^2)$.

- 3 pts d) Compute the H -field produced by the upper surface of the disk at the observation point $z = z_0$ (located on the z -axis) by integrating over the magnetic surface charges of the upper disk.
- 2 pts e) Observing that the lower disk surface (located at $z = -\tau$) has a magnetic charge density $-\sigma_0^{(m)}$ at a distance of $z_0 + \tau$ from the observation point, compute the total H -field produced by both surfaces of the magnetized disk at the observation point $z = z_0$.
- 1 pt f) For sufficiently small values of τ , simplify the expression of $H(z_0)$ for $z_0 > 0$ (and similarly for $z_0 < -\tau$) by invoking the standard differentiation formula $f'(z) \cong [f(z + \tau) - f(z)]/\tau$.

Hint: $\int r dr / (z_0^2 + r^2)^{3/2} = -1/\sqrt{z_0^2 + r^2}$.

Problem 3) Consider a hollow spherical shell of radius R centered at the origin of the xyz coordinate system, having a uniform surface charge density σ_0 . The top cap of the sphere is removed, so that, for the remaining part of the hollow shell, the polar angle θ ranges from θ_0 to π , as shown in the figure.



- 4 pts a) Find the on-axis electric field $\mathbf{E}(\mathbf{r})$ at the axial point $z = z_0$. (**Note:** z_0 can assume any value from $-\infty$ to ∞ .)
- 2 pts b) Confirm that your solution satisfies Maxwell's first boundary condition at the bottom of the spherical shell, where $z_0 = -R$.
- 2 pts c) Verify that, for the complete spherical shell, where $\theta_0 \rightarrow 0$, your solution yields zero for the E -field inside, and $\mathbf{E}(\mathbf{r}) = \pm 4\pi R^2 \sigma_0 \hat{\mathbf{z}} / (4\pi \epsilon_0 z_0^2)$ outside the shell ($\pm$ for $z_0 \gtrless 0$).

Hint: You may want to change the integration variable θ to $x = -\cos \theta$, then use the indefinite integrals

$$\int \frac{dx}{\sqrt{(a+bx)^3}} = -\frac{2}{b\sqrt{a+bx}} \quad \text{and} \quad \int \frac{x dx}{\sqrt{(a+bx)^3}} = \frac{2(2a+bx)}{b^2\sqrt{a+bx}}.$$