

Problem 1) a) According to the second law of motion, the units of force are [newton] = [kg · m/sec²]. The units of the E -field are [volt/m] and those of the B -field are [weber/m²]. The Lorentz force law then yields

$$[\text{newton}] = [\text{kg} \cdot \text{m}/\text{sec}^2] = [\text{coulomb} \cdot \text{volt}/\text{m}] = [\text{coulomb} \cdot (\text{m}/\text{sec}) \cdot (\text{weber}/\text{m}^2)].$$

Given that [coulomb] = [ampère · sec], we find [volt] = [kg · m²/(ampère · sec³)]. Similarly, [weber] = [kg · m²/(ampère · sec²)].

b) In the Ampèrian model, $\mathbf{m} = \mu_0 I A \hat{\mathbf{n}}$ has the units of [(henry/m) · ampère · m²] = [henry · ampère · m]. According to the identity $\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$, [weber/m²] = [(henry/m) · (ampère/m)] and, therefore, [weber] = [henry · ampère]. Consequently, the units of $\mathbf{m} = \mu_0 I A \hat{\mathbf{n}}$ are [weber · meter], matching the units obtained from the alternative definition $\mathbf{m} = q^{(m)} \mathbf{d}$, where the magnetic charge $q^{(m)}$ has units of [weber] and the separation distance $\mathbf{d}$ between the pair of magnetic monopoles has units of [meter].

c) Since $Q = CV$, we have

$$[\text{farad}] = [\text{coulomb}/\text{volt}] = [\text{ampère} \cdot \text{sec}]/[\text{kg} \cdot \text{m}^2/(\text{ampère} \cdot \text{sec}^3)] = [\text{ampère}^2 \cdot \text{sec}^4/(\text{kg} \cdot \text{m}^2)].$$

Similarly, $\Phi = LI$ indicates that [henry] = [weber/ampère] = [kg · m²/(ampère² · sec²)].

Problem 2) a) The magnetic dipole moment of the disk is $\mathbf{m} = \pi R^2 \sigma_0^{(m)} \tau \hat{\mathbf{z}}$. Normalization by the disk volume $\pi R^2 \tau$ yields $\mathbf{M} = \sigma_0^{(m)} \hat{\mathbf{z}}$.

b) $\mathbf{m} = \mu_0 I A \hat{\mathbf{n}} = \mu_0 (J_{s0} \tau) (\pi R^2) \hat{\mathbf{z}}$. Normalization by the disk volume $\pi R^2 \tau$ yields $\mathbf{M} = \mu_0 J_{s0} \hat{\mathbf{z}}$. Overall, we now have $\mathbf{M} = \mu_0 J_{s0} = \sigma_0^{(m)}$.

c) The Coulomb law, derived from Maxwell's 1st equation, is $\mathbf{E}(\mathbf{r}) = q^{(e)} \hat{\mathbf{r}}/(4\pi\epsilon_0 r^2)$. The similarity of Maxwell's 1st and 4th equations now implies that $\mathbf{H}(\mathbf{r}) = q^{(m)} \hat{\mathbf{r}}/(4\pi\mu_0 r^2)$.

d) On the upper disk surface, a thin ring of radius r and width dr has a total magnetic charge of $\sigma_0^{(m)} 2\pi r dr$ and a distance of $\sqrt{r^2 + z^2}$ to any point z on the central z -axis. The symmetry of the ring around the central axis indicates that the H -field's component along the z -axis is the only component that survives integration around the ring. Projection onto the z -axis thus requires multiplication by the cosine of the angle subtended by the ring at the observation point, namely, $\cos \psi = z/\sqrt{r^2 + z^2}$. All in all, we have

$$\begin{aligned} H_z(x=0, y=0, z) &= \int_{r=0}^R \frac{\sigma_0^{(m)} (2\pi r) \cos \psi}{4\pi\mu_0 (r^2 + z^2)} dr = \int_{r=0}^R \frac{\sigma_0^{(m)} (2\pi r) z}{4\pi\mu_0 (r^2 + z^2)^{3/2}} dr \\ &= -\left(\frac{\sigma_0^{(m)} z}{2\mu_0}\right) (r^2 + z^2)^{-1/2} \Big|_{r=0}^R = \left(\frac{\sigma_0^{(m)} z}{2\mu_0}\right) \left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2 + z^2}}\right) \\ &= \frac{\sigma_0^{(m)}}{2\mu_0} \left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + z^2}}\right). \end{aligned} \quad (1)$$

e) For $z_0 > 0$, we have $z_0/|z_0| = 1$. The expression of H_z for the lower disk surface is the same as that for the upper surface, except that $\sigma_0^{(m)}$ is replaced by $-\sigma_0^{(m)}$, and z is replaced by $z + \tau$. The total field at $z = z_0 > 0$ is thus given by

$$\begin{aligned}
H_z(z = z_0) &= \frac{\sigma_0^{(m)}}{2\mu_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right)_{z=z_0} - \frac{\sigma_0^{(m)}}{2\mu_0} \left(1 - \frac{z+\tau}{\sqrt{R^2 + (z+\tau)^2}}\right)_{z=z_0} \\
&= \frac{\sigma_0^{(m)}}{2\mu_0} \left(\frac{z+\tau}{\sqrt{R^2 + (z+\tau)^2}} - \frac{z}{\sqrt{R^2 + z^2}}\right)_{z=z_0}.
\end{aligned} \tag{2}$$

The same expression also holds below the disk in the region $z_0 < -\tau$, where, in Eq.(1), $z/|z| = -1$ for both surfaces of the disk, and the total H -field in Eq.(2) is parallel to the z -axis in both regions above and below the disk.

Inside the disk, where $-\tau < z_0 < 0$, Eq.(2) continues to hold, except that now $z/|z|$ in Eq.(1) equals -1 for the upper disk surface, and $+1$ for the lower disk surface. This results in an added term $-\sigma_0^{(m)}\hat{\mathbf{z}}/\mu_0$ to the total H -field of Eq.(2). This additional term, when multiplied by μ_0 and incremented by $\mathbf{M}$ (in accordance with the identity $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$), makes zero contribution to the B -field inside the disk. Thus, the total B -field is simply μ_0 times the H -field of Eq.(2) at all points z_0 along the z -axis.

f) For sufficiently small values of τ , the expression of $H_z(z = z_0)$ in Eq.(2) can be further simplified, as follows:

$$H_z(z = z_0) = \left(\frac{\sigma_0^{(m)}\tau}{2\mu_0}\right) \frac{d}{dz} \left(\frac{z}{\sqrt{R^2 + z^2}}\right)_{z=z_0} = \left(\frac{\sigma_0^{(m)}\tau}{2\mu_0}\right) \left[\frac{\sqrt{R^2 + z^2} - (z^2/\sqrt{R^2 + z^2})}{R^2 + z^2}\right]_{z=z_0} = \frac{\sigma_0^{(m)}R^2\tau}{2\mu_0(R^2 + z_0^2)^{3/2}}. \tag{3}$$

Note that the above expression is valid both above and below the disk, i.e., whether $z_0 > 0$ or $z_0 < -\tau$, simply because, in arriving at Eq.(3), τ has been taken to be infinitesimally small.

If we substitute J_{s0} for $\sigma_0^{(m)}/\mu_0$ in either Eq.(2) or Eq.(3), then multiply H_z by μ_0 , we will have expressions for the axial B -field of the current loop depicted in figure (b).

Problem 3) a) Symmetry dictates that, at all points along the z -axis, the E -field is either parallel or anti-parallel to the z -axis. For a charged ring of radius $R \sin \theta$, the distance from any point on the ring to the observation point at $z = z_0$ is $\sqrt{(z_0 - R \cos \theta)^2 + (R \sin \theta)^2} = \sqrt{R^2 - 2Rz_0 \cos \theta + z_0^2}$. The corresponding angle ψ , subtended by the ring at the observation point, is then given by

$$\cos \psi = \frac{z_0 - R \cos \theta}{\sqrt{(z_0 - R \cos \theta)^2 + (R \sin \theta)^2}}. \tag{1}$$

The magnitude of the E -field at the observation point is proportional to the charge of the ring, $\sigma_0(2\pi R \sin \theta)Rd\theta$, the squared inverse of the distance from the perimeter of the ring to $z = z_0$, and $\cos \psi$ (accounting for the projection of the E -field onto the z -axis). We thus have

$$\begin{aligned}
E_z &= \int_{\theta=\theta_0}^{\pi} \frac{\sigma_0(2\pi R \sin \theta) \cos \psi}{4\pi\epsilon_0[(z-R \cos \theta)^2 + (R \sin \theta)^2]} R d\theta = \frac{\sigma_0 R^2}{2\epsilon_0} \int_{\theta=\theta_0}^{\pi} \frac{\sin \theta (z - R \cos \theta)}{(R^2 - 2zR \cos \theta + z^2)^{3/2}} d\theta \quad \leftarrow x = -\cos \theta \\
&= \frac{\sigma_0}{2\epsilon_0} \int_{x=-\cos \theta_0}^1 \frac{\zeta + x}{\sqrt{(\zeta^2 + 2\zeta x + 1)^3}} dx \quad \leftarrow \text{Gradshteyn \& Ryzhik 2.223-1, 2} \quad \downarrow \quad = \frac{\sigma_0}{2\epsilon_0} \left[-\frac{2\zeta}{2\zeta\sqrt{\zeta^2 + 2\zeta x + 1}} + \frac{2(2\zeta^2 + 2 + 2\zeta x)}{4\zeta^2\sqrt{\zeta^2 + 2\zeta x + 1}} \right]_{x=-\cos \theta_0}^1 \quad \leftarrow \zeta = z/R \\
&= \frac{\sigma_0}{2\epsilon_0\zeta^2} \left(\frac{1 + \zeta x}{\sqrt{\zeta^2 + 2\zeta x + 1}} \right)_{x=-\cos \theta_0}^1 = \frac{\sigma_0}{2\epsilon_0\zeta^2} \left(\frac{1 + \zeta}{|1 + \zeta|} - \frac{1 - \zeta \cos \theta_0}{\sqrt{\zeta^2 - 2\zeta \cos \theta_0 + 1}} \right).
\end{aligned} \tag{2}$$

b) At $z = -R$, where $\zeta = -1$, the first bracketed term in the above expression of E_z becomes indeterminate. This term, however, equals $+1$ when z is slightly above $-R$, and -1 when z is slightly below $-R$, exhibiting a jump (or discontinuity) equal to 2. In contrast, the second term remains continuous at $z = -R$, its value varying smoothly through $\sqrt{(1 + \cos \theta_0)/2} = \cos(\theta_0/2)$. Thus, the discontinuity of E_z at $z = -R$ is equal to σ_0/ϵ_0 , in agreement with Maxwell's first boundary condition.

c) For the full spherical shell, $\theta_0 = 0$ and, therefore,

$$E_z(z = z_0) = \frac{\sigma_0}{2\epsilon_0\zeta^2} \left(\frac{1+\zeta}{|1+\zeta|} - \frac{1-\zeta}{|1-\zeta|} \right) = \frac{R^2\sigma_0}{2\epsilon_0 z_0^2} \left(\frac{R+z_0}{|R+z_0|} - \frac{R-z_0}{|R-z_0|} \right). \quad (3)$$

Inside the shell, where $-R < z_0 < R$, both absolute values in the above expression of E_z are positive and, therefore, $E_z = 0$. In contrast, when z_0 is outside the shell, the bracketed term is either $+2$ or -2 , yielding $E_z = \pm R^2\sigma_0/(\epsilon_0 z_0^2)$, as expected from a uniformly charged shell.
