**Day 1 Problem**) a) In free space, Maxwell's first equation is $\boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E})=0$. Application to the $E$-field of the plane wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{E}_0=0$. This is the general relation between the $k$-vector and the magnitude $\boldsymbol{E}_0$ of the plane-wave's $E$-field.

b) In free space, Maxwell's fourth equation is $\boldsymbol{\nabla}\cdot(\mu_0\boldsymbol{H})=0$. Application to the $H$-field of the plane wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. This is the general relation between the $k$-vector and the magnitude $\boldsymbol{H}_0$ of the plane-wave's $H$-field.

c) Maxwell's second equation in free space is $\boldsymbol{\nabla}\times\boldsymbol{H}=\varepsilon_0\,\partial\boldsymbol{E}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=-\mathrm{i}\omega\varepsilon_0\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{H}_0=-\varepsilon_0\omega\boldsymbol{E}_0.$$

d) Maxwell's third equation in free space is $\boldsymbol{\nabla}\times\boldsymbol{E}=-\mu_0\,\partial\boldsymbol{H}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=\mathrm{i}\omega\mu_0\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{E}_0=\mu_0\omega\boldsymbol{H}_0.$$

e) From part (c), we know that $\boldsymbol{E}_0=-\boldsymbol{k}\times\boldsymbol{H}_0/(\varepsilon_0\omega)$. Substitution in the result obtained in part (d) then yields

$$-\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H}_0)/(\varepsilon_0\omega)=\mu_0\omega\boldsymbol{H}_0\qquad\rightarrow\qquad(\boldsymbol{k}\cdot\boldsymbol{H}_0)\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=-\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0.$$

Now, in part (b) we found that $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. Therefore, the first term on the left-hand side of the preceding equation disappears, and we are left with $(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0$. Dropping $\boldsymbol{H}_0$ from both sides of this equation yields

$$\boldsymbol{k}\cdot\boldsymbol{k}=(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=(k'^2-k''^2)+2\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{k}''=\mu_0\varepsilon_0\omega^2=(\omega/c)^2.$$

This is the general relation between the wave-vector $\boldsymbol{k}$ and the frequency $\omega$ of a plane wave in free space.

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**Day 2 Problem)**

a) $\boldsymbol{E}(z,t) = |E_{x0}|\cos(k_0 z - \omega t + \varphi_0)\,\hat{\boldsymbol{x}}$ and $\boldsymbol{H}(z,t) = (|E_{x0}|/Z_0)\cos(k_0 z - \omega t + \varphi_0)\,\hat{\boldsymbol{y}}$. Note that $\boldsymbol{E}$ and $\boldsymbol{H}$ have the same phase $\varphi_0$.

$\boldsymbol{E}'(z,t) = |E_{x0}|\cos(k_0 z + \omega t + \varphi_0')\,\hat{\boldsymbol{x}}$ and $\boldsymbol{H}'(z,t) = -(|E_{x0}|/Z_0)\cos(k_0 z + \omega t + \varphi_0')\,\hat{\boldsymbol{y}}$. Again, $\boldsymbol{E}'$ and $\boldsymbol{H}'$ have the same phase $\varphi_0'$, although it could differ from $\varphi_0$.

Total $E$-field: $\boldsymbol{E}(z,t) + \boldsymbol{E}'(z,t) = |E_{x0}|[\cos(k_0 z - \omega t + \varphi_0) + \cos(k_0 z + \omega t + \varphi_0')]\hat{\boldsymbol{x}}$

$$= 2|E_{x0}|\cos[k_0 z + ½(\varphi_0' + \varphi_0)]\cos[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{x}}.$$

Total $H$-field: $\boldsymbol{H}(z,t) + \boldsymbol{H}'(z,t) = (|E_{x0}|/Z_0)[\cos(k_0 z - \omega t + \varphi_0) - \cos(k_0 z + \omega t + \varphi_0')]\hat{\boldsymbol{y}}$

$$= 2(|E_{x0}|/Z_0)\sin[k_0 z + ½(\varphi_0' + \varphi_0)]\sin[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{y}}.$$

Since the total $E$-field at the mirrors' inward-facing surfaces must vanish, we have

First mirror surface at $z = 0$: $\quad 2|E_{x0}|\cos[½(\varphi_0' + \varphi_0)]\cos[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{x}} = 0.$

Second mirror surface at $z = d$: $\quad 2|E_{x0}|\cos[k_0 d + ½(\varphi_0' + \varphi_0)]\cos[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{x}} = 0.$

Consequently,

$$\begin{cases}\cos[½(\varphi_0' + \varphi_0)] = 0 \\ \cos[k_0 d + ½(\varphi_0' + \varphi_0)] = 0\end{cases} \rightarrow \begin{cases}\varphi_0 + \varphi_0' = (2n+1)\pi & \text{(odd integer multiple of } \pi\text{);} \\ k_0 d = m\pi \ \rightarrow\ d = m\lambda_0/2 & \text{(integer multiple of } \lambda_0/2\text{).}\end{cases}$$

The total fields in the cavity are thus found to be

$$\boldsymbol{E}(z,t) = \mp 2|E_{x0}|\sin(k_0 z)\cos[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{x}}.$$

$$\boldsymbol{H}(z,t) = \pm 2(|E_{x0}|/Z_0)\cos(k_0 z)\sin[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{y}}.$$

**Note**: $\cos[k_0 z + ½(\varphi_0' + \varphi_0)] = \cos(k_0 z)\cos[½(\varphi_0' + \varphi_0)] - \sin(k_0 z)\sin[½(\varphi_0' + \varphi_0)]$, which reduces to $-\sin(k_0 z)\sin[(n + ½)\pi]$ and, depending on the value of $n$, simplifies to $\mp\sin(k_0 z)$. In similar fashion, $\sin[k_0 z + ½(\varphi_0' + \varphi_0)] = \sin(k_0 z)\cos[½(\varphi_0' + \varphi_0)] + \cos(k_0 z)\sin[½(\varphi_0' + \varphi_0)]$, which reduces to $\cos(k_0 z)\sin[(n + ½)\pi]$ and, depending on the value of $n$, simplifies to $\pm\cos(k_0 z)$.

b) The surface-current-density equals the total tangential $H$-field at each mirror surface. For the first mirror at $z = 0$, $\cos(k_0 z) = 1$. For the second mirror at $z = d$, $\cos(k_0 z) = \cos(m\pi) = \pm 1$. The magnitude of the surface-current-density on both mirrors is, therefore, $J_{so} = 2|E_{x0}|/Z_0$.

c) The trapped energy per unit cross-sectional area is given by

Trapped $E$-field energy:

$$\mathcal{E}_{\text{electric}} = ½\varepsilon_0\int_0^d |\boldsymbol{E}(z,t)|^2 \mathrm{d}z = 2\varepsilon_0|E_{x0}|^2\cos^2[\omega t + ½(\varphi_0' - \varphi_0)]\int_0^d \sin^2(k_0 z)\,\mathrm{d}z$$

$$= \varepsilon_0|E_{x0}|^2 d\cos^2[\omega t + ½(\varphi_0' - \varphi_0)].$$

Trapped $H$-field energy:

$$\mathcal{E}_{\text{magnetic}} = ½\mu_0\int_0^d |\boldsymbol{H}(z,t)|^2 \mathrm{d}z = 2\mu_0(|E_{x0}|/Z_0)^2\sin^2[\omega t + ½(\varphi_0' - \varphi_0)]\int_0^d \cos^2(k_0 z)\,\mathrm{d}z$$

$$= \varepsilon_0|E_{x0}|^2 d\sin^2[\omega t + ½(\varphi_0' - \varphi_0)].$$

The peak values of the $E$-field and $H$-field energies (per unit cross-sectional area) are thus equal to $\varepsilon_0|E_{x0}|^2 d$. However, there exists a phase difference between these two entities: When the $E$-field energy is zero, the $H$-field energy is at a maximum, and vice-versa. At one instant, all the energy is in the $E$-field; a quarter of a period later, all the energy is in the $H$-field. The energy thus swings back and forth from one form to the other.

d) $\boldsymbol{S}(z,t) = \boldsymbol{E}(z,t) \times \boldsymbol{H}(z,t)$

$$= -(4|E_{x0}|^2/Z_0)\sin(k_0 z)\cos(k_0 z)\sin[\omega t + ½(\varphi_0' - \varphi_0)]\cos[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{z}}$$

$$= -(|E_{x0}|^2/Z_0)\sin(2k_0 z)\sin[2\omega t + (\varphi_0' - \varphi_0)]\hat{\boldsymbol{z}}.$$

At the nodes of the $E$-field, as well as those of the $H$-field, the Poynting vector is zero. No energy, therefore, crosses these nodes. In between the nodes, the energy flows to the right for one quarter of one oscillation period ($T = 2\pi/\omega$), then flows to the left during the next quarter. The process is then repeated.