Opti 501 **Final Exam** (12/16/2025) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

13 pts **Problem 1**) The Lorentz oscillator model relates the local polarization $\boldsymbol{P}(t)$ at any given point $\boldsymbol{r}$ within a linear, isotropic medium to the electric field $\boldsymbol{E}(t)$ at the same point $\boldsymbol{r}$ via the following second-order ordinary differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{P}(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}(t) + \omega_0^2\boldsymbol{P}(t) = \varepsilon_0\omega_p^2\boldsymbol{E}(t).$$

Here, γ is the damping coefficient, ω_0 the resonance frequency, and ω_p the plasma frequency of the host medium. In this problem you are asked to solve the above equation for $\boldsymbol{P}(t)$ *without* using the complex notation. This requires that you write $\boldsymbol{E}(t) = \boldsymbol{E}_0'\cos(\omega t) + \boldsymbol{E}_0''\sin(\omega t)$, where $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are arbitrary real-valued vectors, while ω represents the oscillation frequency of the exciting electric field. You may begin by assuming the following form for the polarization:

$$\boldsymbol{P}(t) = \boldsymbol{P}_0'\cos(\omega t + \varphi_0') + \boldsymbol{P}_0''\sin(\omega t + \varphi_0''),$$

then solve the differential equation to find expressions for $\boldsymbol{P}_0'$, φ_0', $\boldsymbol{P}_0''$, and φ_0''. Your final result should be the same as that obtained in the textbook with the aid of complex notation.

Hint: $\cos a\cos b - \sin a\sin b = \cos(a+b)$ and $\sin a\cos b + \cos a\sin b = \sin(a+b)$.

Problem 2) The Fresnel reflection coefficients for p- and s-polarized light at the interfacial xy-plane between the incidence medium (a) and the transmittance medium (b) are given by

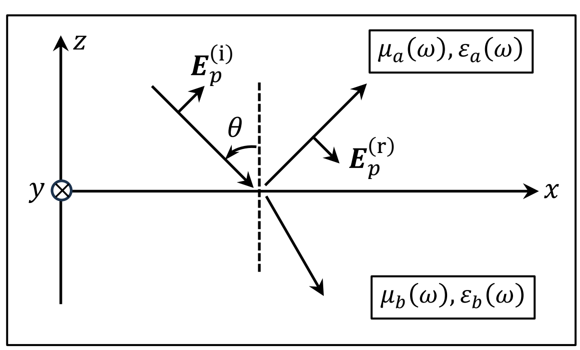
$$\rho_p = E_{x0}^{(\mathrm{r})}/E_{x0}^{(\mathrm{i})} = \frac{\varepsilon_a\sqrt{\mu_b\varepsilon_b-(ck_x/\omega)^2} - \varepsilon_b\sqrt{\mu_a\varepsilon_a-(ck_x/\omega)^2}}{\varepsilon_a\sqrt{\mu_b\varepsilon_b-(ck_x/\omega)^2} + \varepsilon_b\sqrt{\mu_a\varepsilon_a-(ck_x/\omega)^2}},$$

$$\rho_s = E_{y0}^{(\mathrm{r})}/E_{y0}^{(\mathrm{i})} = \frac{\mu_b\sqrt{\mu_a\varepsilon_a-(ck_x/\omega)^2} - \mu_a\sqrt{\mu_b\varepsilon_b-(ck_x/\omega)^2}}{\mu_b\sqrt{\mu_a\varepsilon_a-(ck_x/\omega)^2} + \mu_a\sqrt{\mu_b\varepsilon_b-(ck_x/\omega)^2}}.$$

It is assumed here that the plane of incidence is xz (i.e., $k_y = 0$), the angle of incidence is θ (see the figure below), the frequency of the incident plane-wave is ω, and the incidence medium's relative permeability $\mu_a(\omega)$ and relative permittivity $\varepsilon_a(\omega)$ are real-valued and positive.

3 pts a) Show that $\rho_p = \rho_s$ at normal incidence (i.e., when $\theta = 0$).

4 pts b) Find ρ_p and ρ_s at grazing incidence (i.e., when $\theta \to 90°$). Recalling that $\tau_p = 1 + \rho_p$ and $\tau_s = 1 + \rho_s$, your results should indicate that $\tau_p \to 2$ and $\tau_s \to 0$ as $\theta \to 90°$. The result for τ_s makes sense, since one expects the beam transmitted into medium b to vanish at grazing incidence. However, $\tau_p \to 2$ is counter-intuitive, since the transmitted beam *appears* to not have vanished at grazing incidence. How do you reconcile the



sensible expectation with the nonzero value of the Fresnel transmission coefficient τ_p at grazing incidence?

3 pts c) Assuming that $\mu_b = \mu_a$ and that ε_b is also real and positive, show that an incidence angle θ exits at which $\rho_p = 0$; this is the so-called Brewster's angle θ_B. Under the circumstances, show that $\theta_B = \arctan(n_b/n_a)$.

4 pts d) Let $\mu_b = \mu_a$, and also let the real and positive ε_b be less than ε_a. In terms of n_a and n_b, what is the expression of θ_c (i.e., the critical angle of total internal reflection, where $|\rho_p| = |\rho_s| = 1$ for $\theta \geq \theta_c$)? Find expressions for the phase angles of ρ_p and ρ_s (as functions of θ and θ_c) when the incidence angle θ is between θ_c and 90°.

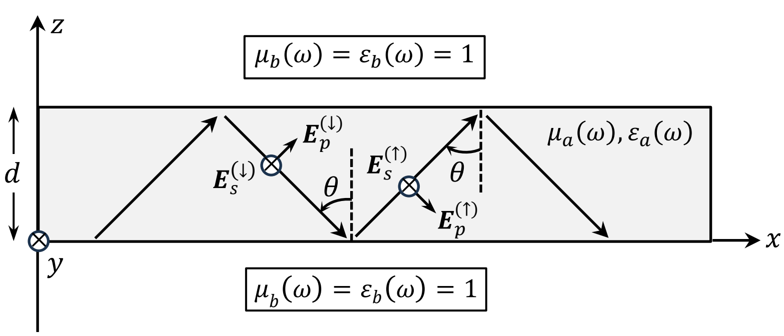
Problem 3) A transparent dielectric slab of thickness d and refractive index $n_a(\omega)$ serves as a waveguide to transport electromagnetic radiation of frequency ω along the x-axis. Consider a pair of plane-waves that are totally internally reflected at the boundaries between the slab and its surrounding medium of refractive index $n_b(\omega) = 1$. Assuming that $\mu_a = \mu_b = 1$, and that the critical angle of total internal reflection is $\theta_c = \arcsin(n_b/n_a)$, the phase angles of the Fresnel reflection coefficients ρ_p and ρ_s at either boundary when $\theta \geq \theta_c$ are given by

$$\varphi_p = \pi - 2\arctan\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{(\sin^2\theta_c)\cos\theta}\right), \qquad \varphi_s = -2\arctan\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta}\right).$$

The downward propagating wave (↓) is reflected at the lower facet of the slab at $z = 0$, then propagates upward (↑) to reach $z = d$, bounces back at the upper facet, then propagates downward (↓) to reach $z = 0$ again. The accumulated phase due to up and down propagation is

$$k_z^{(\uparrow)} d + k_z^{(\downarrow)}(-d) = 2d(\omega/c)n_a(\omega)\cos\theta.$$

To this, one must add the acquired phase upon total internal reflection at the slab's facets (i.e., $2\varphi_p$ for p-light, or $2\varphi_s$ for s-light) in order to determine the total phase acquired in each roundtrip. Self-consistency then demands that the total accumulated phase in each roundtrip be an integer-multiple of 2π.



8 pts a) The self-consistency requirement restricts the allowed incidence angle θ inside the slab waveguide. Describe a graphical method that one could use to compute the allowed angles θ.

5 pts b) Each allowed angle θ corresponds to a different propagation mode within the waveguide. Explain why, in general, the waveguide modes for p-light differ from those for s-light.

Hint: Treat p-light and s-light separately. In each case, denote by $2\pi m$ an arbitrary integer-multiple of 2π. The acceptable values of the incidence angle θ in each case will depend on whether m is odd or even.
