Problem 1) Substituting the suggested solution for $\boldsymbol{P}(t)$ in the governing differential equation, we arrive at

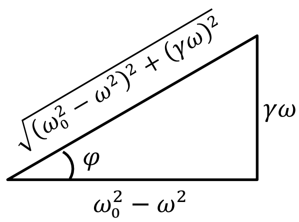
$$-\omega^2\boldsymbol{P}_0'\cos(\omega t+\varphi_0')-\omega^2\boldsymbol{P}_0''\sin(\omega t+\varphi_0'')-\gamma\omega\boldsymbol{P}_0'\sin(\omega t+\varphi_0')+\gamma\omega\boldsymbol{P}_0''\cos(\omega t+\varphi_0'')$$

$$+\omega_0^2\boldsymbol{P}_0'\cos(\omega t+\varphi_0')+\omega_0^2\boldsymbol{P}_0''\sin(\omega t+\varphi_0'')=\varepsilon_0\omega_p^2\boldsymbol{E}_0'\cos(\omega t)+\varepsilon_0\omega_p^2\boldsymbol{E}_0''\sin(\omega t).$$

The above equation may now be split into two, one for $\boldsymbol{P}_0'$ and φ_0', the other for $\boldsymbol{P}_0''$ and φ_0'', as follows:

$$[(\omega_0^2-\omega^2)\cos(\omega t+\varphi_0')-\gamma\omega\sin(\omega t+\varphi_0')]\boldsymbol{P}_0'=\varepsilon_0\omega_p^2\boldsymbol{E}_0'\cos(\omega t),$$

$$[(\omega_0^2-\omega^2)\sin(\omega t+\varphi_0'')+\gamma\omega\cos(\omega t+\varphi_0'')]\boldsymbol{P}_0''=\varepsilon_0\omega_p^2\boldsymbol{E}_0''\sin(\omega t).$$



$0\le\omega\le\omega_0 \;\rightarrow\; 0\le\varphi\le 90°,$
$\omega>\omega_0 \;\rightarrow\; 90°<\varphi<180°.$

In the right triangle depicted above, the length of the hypotenuse is $\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}$ and the angle φ is $\arctan[\gamma\omega/(\omega_0^2-\omega^2)]$. The above equations may thus be written as

$$\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}[\cos(\varphi)\cos(\omega t+\varphi_0')-\sin(\varphi)\sin(\omega t+\varphi_0')]\boldsymbol{P}_0'=\varepsilon_0\omega_p^2\boldsymbol{E}_0'\cos(\omega t),$$

$$\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}[\cos(\varphi)\sin(\omega t+\varphi_0'')+\sin(\varphi)\cos(\omega t+\varphi_0'')]\boldsymbol{P}_0''=\varepsilon_0\omega_p^2\boldsymbol{E}_0''\sin(\omega t).$$

The equations are further simplified with the aid of standard trigonometric identities, as follows:

$$\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}\cos(\omega t+\varphi_0'+\varphi)\,\boldsymbol{P}_0'=\varepsilon_0\omega_p^2\boldsymbol{E}_0'\cos(\omega t),$$

$$\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}\sin(\omega t+\varphi_0''+\varphi)\,\boldsymbol{P}_0''=\varepsilon_0\omega_p^2\boldsymbol{E}_0''\sin(\omega t).$$

It is thus seen that $\varphi_0'=\varphi_0''=-\varphi$ and that, therefore,

$$\boldsymbol{P}(t)=\frac{\varepsilon_0\omega_p^2}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{½}}[\boldsymbol{E}_0'\cos(\omega t-\varphi)+\boldsymbol{E}_0''\sin(\omega t-\varphi)].$$

The above solution is in agreement with that obtained in the textbook using complex notation, as shown below.

$$\boldsymbol{P}(t)=\text{Re}\left[\frac{\varepsilon_0\omega_p^2}{\omega_0^2-\omega^2-\mathrm{i}\gamma\omega}\boldsymbol{E}_0e^{-\mathrm{i}\omega t}\right]=\text{Re}\left\{\frac{\varepsilon_0\omega_p^2}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{½}\,e^{-\mathrm{i}\varphi}}(\boldsymbol{E}_0'+\mathrm{i}\boldsymbol{E}_0'')e^{-\mathrm{i}\omega t}\right\}$$

$\varphi=\arctan[\gamma\omega/(\omega_0^2-\omega^2)]$

$$=\frac{\varepsilon_0\omega_p^2}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{½}}\text{Re}\{(\boldsymbol{E}_0'+\mathrm{i}\boldsymbol{E}_0'')[\cos(\omega t-\varphi)-\mathrm{i}\sin(\omega t-\varphi)]\}$$

$$=\frac{\varepsilon_0\omega_p^2}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{½}}[\boldsymbol{E}_0'\cos(\omega t-\varphi)+\boldsymbol{E}_0''\sin(\omega t-\varphi)].$$

Problem 2) a) At normal incidence, we have $\theta = 0$ and, therefore, $k_x = (\omega/c)n_a \sin\theta = 0$. We will have

divide numerator and denominator by $\sqrt{\mu_a\varepsilon_a\mu_b\varepsilon_b}$

$$\rho_p = \frac{\varepsilon_a\sqrt{\mu_b\varepsilon_b} - \varepsilon_b\sqrt{\mu_a\varepsilon_a}}{\varepsilon_a\sqrt{\mu_b\varepsilon_b} + \varepsilon_b\sqrt{\mu_a\varepsilon_a}} = \frac{\sqrt{\varepsilon_a/\mu_a} - \sqrt{\varepsilon_b/\mu_b}}{\sqrt{\varepsilon_a/\mu_a} + \sqrt{\varepsilon_b/\mu_b}},$$

divide numerator and denominator by $\mu_a\mu_b$

$$\rho_s = \frac{\mu_b\sqrt{\mu_a\varepsilon_a} - \mu_a\sqrt{\mu_b\varepsilon_b}}{\mu_b\sqrt{\mu_a\varepsilon_a} + \mu_a\sqrt{\mu_b\varepsilon_b}} = \frac{\sqrt{\varepsilon_a/\mu_a} - \sqrt{\varepsilon_b/\mu_b}}{\sqrt{\varepsilon_a/\mu_a} + \sqrt{\varepsilon_b/\mu_b}}.$$

It is seen that $\rho_p = \rho_s$ at $\theta = 0$. (Alternatively, you may start by setting $k_x = 0$, then write $\rho_p = A/B$ and $\rho_s = C/D$, then proceed to verify that $AD = BC$.)

b) At grazing incidence, where $\theta = 90°$, we have $k_x = (\omega/c)n_a \sin\theta = (\omega/c)n_a$. Consequently,

$$\rho_p = \frac{E_{x0}^{(\mathrm{r})}}{E_{x0}^{(\mathrm{i})}} = \frac{\varepsilon_a\sqrt{\mu_b\varepsilon_b - \mu_a\varepsilon_a}}{\varepsilon_a\sqrt{\mu_b\varepsilon_b - \mu_a\varepsilon_a}} = 1,$$

$$\rho_s = \frac{E_{y0}^{(\mathrm{r})}}{E_{y0}^{(\mathrm{i})}} = -\frac{\mu_a\sqrt{\mu_b\varepsilon_b - \mu_a\varepsilon_a}}{\mu_a\sqrt{\mu_b\varepsilon_b - \mu_a\varepsilon_a}} = -1.$$

In the case of s-polarization, we find $\tau_s = 1 + \rho_s = 0$, which indicates the vanishing of the transmitted beam, as expected. In the immediate vicinity of the interfacial xy-plane at $z = 0^+$, the reflected beam propagates parallel to the incident beam, albeit with its y-oriented E-field flipped to the $-y$ direction. Similarly, the reflected beam's H-field has flipped from $+z$ to $-z$ orientation. Thus, there are no electric and magnetic fields parallel to the interface, nor any such fields perpendicular to the interfacial plane.

In the case of p-polarization, the Fresnel reflection coefficient represents the E-field's amplitude along the x-axis. However, at grazing incidence, E_x is essentially zero, for both the incident and reflected beams. The equality of the incident and reflected E_x (i.e., the fact that ρ_p approaches 1.0 as $\theta \to 90°$) thus indicates the equality of the incident and reflected E_z, albeit with a sign change, as can be inferred from the provided figure in the statement of the problem. In other words, it is the reflection coefficient for E_z that approaches -1 as $\theta \to 90°$. The H-field (always aligned with the y-axis for p-light) similarly flips upon reflection. All in all, the total E_z and the total H_y immediately above the xy-plane at $z = 0^+$ vanish. Consequently, the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the transmitted plane-wave immediately below the interface at $z = 0^-$ approach zero as $\theta \to 90°$.

c) $\rho_p = 0 \quad \rightarrow \quad \varepsilon_a\sqrt{\mu_b\varepsilon_b - (ck_x/\omega)^2} = \varepsilon_b\sqrt{\mu_a\varepsilon_a - (ck_x/\omega)^2}$

$$\rightarrow \quad \varepsilon_a^2[\mu_b\varepsilon_b - (n_a\sin\theta)^2] = \varepsilon_b^2[\mu_a\varepsilon_a - (n_a\sin\theta)^2]$$

$$\rightarrow \quad \varepsilon_a^2\mu_b\varepsilon_b - \varepsilon_a^2\mu_a\varepsilon_a\sin^2\theta = \varepsilon_b^2\mu_a\varepsilon_a - \varepsilon_b^2\mu_a\varepsilon_a\sin^2\theta$$

$$\rightarrow \quad \sin^2\theta = \frac{(\varepsilon_a\mu_b - \varepsilon_b\mu_a)\varepsilon_b}{\mu_a(\varepsilon_a^2 - \varepsilon_b^2)}$$

$$\rightarrow \quad \cos^2\theta = 1 - \sin^2\theta = \frac{(\mu_a\varepsilon_a - \mu_b\varepsilon_b)\varepsilon_a}{\mu_a(\varepsilon_a^2 - \varepsilon_b^2)}$$

$$\rightarrow \quad \tan^2\theta = \frac{(\mu_b\varepsilon_a - \mu_a\varepsilon_b)\varepsilon_b}{(\mu_a\varepsilon_a - \mu_b\varepsilon_b)\varepsilon_a}.$$

If $\mu_b = \mu_a$, we will have $\tan^2\theta = \mu_b\varepsilon_b/\mu_a\varepsilon_a = (n_b/n_a)^2$. Now, if ε_b happens to be real and positive, there will exist a solution for θ. The Brewster angle will then be $\theta_B = \tan^{-1}(n_b/n_a)$.

Digression. To explore the situations in which a Brewster's angle might exist for s-polarized light, we set ρ_s to zero, then examine the conditions under which a solution for θ might exist.

$$\rho_s = 0 \quad \rightarrow \quad \mu_b\sqrt{\mu_a\varepsilon_a - (ck_x/\omega)^2} = \mu_a\sqrt{\mu_b\varepsilon_b - (ck_x/\omega)^2}$$

$$\rightarrow \quad \mu_b^2\mu_a\varepsilon_a(1-\sin^2\theta) = \mu_a^2(\mu_b\varepsilon_b - \mu_a\varepsilon_a\sin^2\theta)$$

$$\rightarrow \quad (\mu_b\varepsilon_a - \mu_a\varepsilon_b)\mu_b = \varepsilon_a(\mu_b^2 - \mu_a^2)\sin^2\theta \quad \rightarrow \quad \sin^2\theta = \frac{(\mu_b\varepsilon_a - \mu_a\varepsilon_b)\mu_b}{(\mu_b^2 - \mu_a^2)\varepsilon_a}.$$

Clearly, it is possible to have a real-valued angle θ that satisfies the above equation — hence the feasibility of a Brewster's angle for s-light — but *not* if $\mu_b = \mu_a$. In other words, $\mu_b \neq \mu_a$ is a necessary condition for the existence of such an angle, provided that the values of $\mu_a, \varepsilon_a, \mu_b, \varepsilon_b$ are such that the above expression for $\sin^2\theta$ turns out to yield a real number in the $(0,1)$ interval.

d) Setting $ck_x/\omega = n_a\sin\theta$, we note that $\sqrt{n_b^2 - (n_a\sin\theta)^2}$ becomes imaginary when $n_a\sin\theta$ exceeds n_b. The critical angle beyond which this change occurs is $\theta_c = \arcsin(n_b/n_a)$. Thus, upon substituting $n_a^2\sin^2(\theta_c)$ for n_b^2, and also setting $\mu_b = \mu_a$, the simplified Fresnel reflection coefficients become

$\varepsilon_b/\varepsilon_a = \sin^2(\theta_c)$

$$\rho_p = \frac{\varepsilon_a(n_a^2\sin^2\theta_c - n_a^2\sin^2\theta)^{½} - \varepsilon_b(n_a^2 - n_a^2\sin^2\theta)^{½}}{\varepsilon_a(n_a^2\sin^2\theta_c - n_a^2\sin^2\theta)^{½} + \varepsilon_b(n_a^2 - n_a^2\sin^2\theta)^{½}} = \frac{\sqrt{\sin^2\theta_c - \sin^2\theta} - (\varepsilon_b/\varepsilon_a)\cos\theta}{\sqrt{\sin^2\theta_c - \sin^2\theta} + (\varepsilon_b/\varepsilon_a)\cos\theta},$$

$$\rho_s = \frac{(n_a^2 - n_a^2\sin^2\theta)^{½} - (n_a^2\sin^2\theta_c - n_a^2\sin^2\theta)^{½}}{(n_a^2 - n_a^2\sin^2\theta)^{½} + (n_a^2\sin^2\theta_c - n_a^2\sin^2\theta)^{½}} = \frac{\cos\theta - \sqrt{\sin^2\theta_c - \sin^2\theta}}{\cos\theta + \sqrt{\sin^2\theta_c - \sin^2\theta}}.$$

At incidence angles $\theta > \theta_c$, the term under the square root in the above expressions of ρ_p and ρ_s becomes negative, in which case we can write

$$\rho_p = \frac{\mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c} - \sin^2\theta_c\cos\theta}{\mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c} + \sin^2\theta_c\cos\theta} = -\frac{\sin^2\theta_c\cos\theta - \mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c}}{\sin^2\theta_c\cos\theta + \mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c}} = -\exp\left[-2\mathrm{i}\tan^{-1}\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{(\sin^2\theta_c)\cos\theta}\right)\right].$$

It is seen that $|\rho_p| = 1$ and $\varphi_p = \pi - 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{(\sin^2\theta_c)\cos\theta}\right)$ for all angles $\theta \geq \theta_c$. Similarly, in the case of s-polarized (or TE) incident light, we have

$$\rho_s = \frac{\cos\theta - \mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta + \mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c}} = \exp\left[-2\mathrm{i}\tan^{-1}\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta}\right)\right].$$

In this case, $|\rho_s| = 1$ and $\varphi_s = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta}\right)$ for all angles $\theta \geq \theta_c$.

Problem 3) a) Considering that $\boldsymbol{k}^{\uparrow\downarrow} = (\omega/c)n_a(\omega)(\sin\theta\,\hat{\boldsymbol{x}} \pm \cos\theta\,\hat{\boldsymbol{z}})$, the sequence starting with the incident E-field at the lower facet (i.e., at $z = 0$), followed by total internal reflection (TIR) at the lower facet, propagation to the upper facet (at $z = d$), TIR at the upper facet, and, finally, a downward propagation to the lower facet is listed below for p-polarized light.

i) $$E_x^{\downarrow}(x, y, z = 0, t) = E_{0x}e^{\mathrm{i}(k_x x + k_z^{\downarrow}z - \omega t)}\Big|_{z=0} = E_{0x}e^{\mathrm{i}(k_x x - \omega t)}, \qquad (1)$$

ii) $$E_x^{\uparrow}(x, y, z = 0, t) = \rho_p E_{0x}e^{\mathrm{i}(k_x x - \omega t)}, \qquad (2)$$

iii) $E_x^{\uparrow}(x,y,z=d,t) = \rho_p E_{0x} e^{\mathrm{i}(k_x x + k_z^{\uparrow} z - \omega t)}\Big|_{z=d} = \rho_p E_{0x} e^{\mathrm{i}(k_x x + k_z^{\uparrow} d - \omega t)},$ (3)

iv) $E_x^{\downarrow}(x,y,z=d,t) = \rho_p^2 E_{0x} e^{\mathrm{i}(k_x x + k_z^{\uparrow} d - \omega t)},$ (4)

v) $E_x^{\downarrow}(x,y,z=0,t) = \rho_p^2 E_{0x} e^{\mathrm{i}(k_x x + k_z^{\uparrow} d - \omega t)} e^{\mathrm{i}k_z^{\downarrow}(-d)} = \rho_p^2 E_{0x} e^{\mathrm{i}(k_x x + 2k_z^{\uparrow} d - \omega t)}.$ (5)

Self-consistency demands that the E-field after one roundtrip coincide with the E-field at the starting point — aside from an integer-multiple of 2π phase-shift. Equating the E-field in step (v) with that in step (i) and accounting for the possible $2\pi m$ phase-shift, we find

$$\rho_p^2 E_{0x} e^{\mathrm{i}(k_x x + 2k_z^{\uparrow} d - \omega t)} = E_{0x} e^{\mathrm{i}(k_x x - \omega t)} e^{\mathrm{i}2\pi m} \quad \rightarrow \quad \rho_p^2 e^{\mathrm{i}2k_z^{\uparrow} d} = e^{\mathrm{i}2\pi m}$$

$$\rightarrow \quad \left(\frac{\mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c} - \sin^2\theta_c \cos\theta}{\mathrm{i}\sqrt{\sin^2\theta - \sin^2\theta_c} + \sin^2\theta_c \cos\theta}\right)^2 e^{\mathrm{i}2(\omega/c)n_a d\cos\theta} = e^{\mathrm{i}2\pi m}$$

$$\rightarrow \quad 2\pi - 4\arctan\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\sin^2\theta_c \cos\theta}\right) + 2(\omega/c)n_a d\cos\theta = 2m\pi$$

$$\rightarrow \quad \arctan\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\sin^2\theta_c \cos\theta}\right) = ½(\omega/c)n_a d\cos\theta - ½(m-1)\pi. \tag{6}$$

Depending on whether m (an integer) is odd or even, Eq.(6) can be further streamlined to yield

$$m \text{ odd:} \qquad \frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\sin^2\theta_c \cos\theta} = \tan[½(\omega/c)n_a d\cos\theta], \tag{7}$$

$$m \text{ even:} \qquad \frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\sin^2\theta_c \cos\theta} = -\cot[½(\omega/c)n_a d\cos\theta]. \tag{8}$$

As for s-polarized light, a similar procedure yields the acceptable values of θ as follows:

$$\arctan\left(\frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta}\right) = ½(\omega/c)n_a d\cos\theta - ½m\pi. \tag{9}$$

Once again, depending on whether m is odd or even, we find

$$m \text{ even:} \qquad \frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta} = \tan[½(\omega/c)n_a d\cos\theta], \tag{10}$$

$$m \text{ odd:} \qquad \frac{\sqrt{\sin^2\theta - \sin^2\theta_c}}{\cos\theta} = -\cot[½(\omega/c)n_a d\cos\theta], \tag{11}$$

Equation (7), (8), (10), and (11) can be solved graphically by plotting the left- and right-hand sides of each equation versus θ (over the interval $\theta_c < \theta < 90°$), then identifying the crossing points of each pair of graphs.

b) The acceptable values of θ are not necessarily the same for p-light and s-light. This is because $\sin^2(\theta_c)$ appears in the denominator on the left-hand sides of Eqs.(7) and (8), but is absent from Eqs.(10) and (11). This indicates that the characteristics of the p-polarized (or TM) modes of a slab waveguide generally differ from those of the s-polarized (or TE) modes, as the modes associated with different polarizations will have different allowed values of θ for any given set of values of $\omega, n_a(\omega)$, and d. Also, for each polarization, the odd and even modes will have different k_x values and different E-field profiles (as well as H-field profiles) along the z-axis.