Opti 501 **1st Midterm Exam** (9/30/2025) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

5 pts **Problem 1**) a) Suppose that magnetic monopoles have been discovered, and that you are asked to modify Maxwell's equations to account for their existence. Let the free *magnetic* charge and current densities be given by $\rho_{\text{free}}^{(m)}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\text{free}}^{(m)}(\boldsymbol{r},t)$, respectively. Modify the relevant equations of Maxwell to incorporate the free magnetic charge and current densities. Verify that your modified equations do in fact satisfy the *magnetic* charge-current continuity equation.

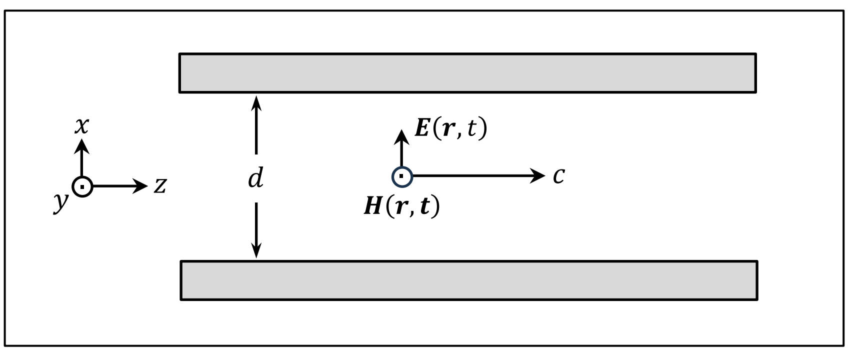
3 pts b) The Poynting vector $\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)$ retains its standard form in the presence of magnetic monopoles. However, the Poynting theorem must now account for the exchange of energy between the electromagnetic fields and the free magnetic current. Derive the modified Poynting theorem, and explain the meaning of the new term in the context of energy conservation.

Problem 2) A plane, monochromatic, linearly-polarized, electromagnetic (EM) wave propagates in the free space region between two perfectly-electrically-conducting parallel plates, as shown in the figure below. The gap-width is d, the plates are infinitely large, and the (real-valued) $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in the gap region are specified as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0 \cos[(\omega/c)z - \omega t]\hat{\boldsymbol{x}},$$

$$\boldsymbol{H}(\boldsymbol{r},t) = (E_0/Z_0) \cos[(\omega/c)z - \omega t]\hat{\boldsymbol{y}}.$$

Here, $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the speed of light in vacuum, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space, and ω is the oscillation frequency of the EM fields.



4 pts a) Show that, in the empty space between the plates, the above $\boldsymbol{E}$ and $\boldsymbol{H}$ fields satisfy all four of Maxwell's equations.

3 pts b) Determine the Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$, the energy-density of the E-field, and the energy-density of the H-field. Show that these entities uphold the law of conservation of energy in the free space region between the plates.

2 pts c) Find the surface charge-density σ_s, and the surface current-density $\boldsymbol{J}_s$, on the interior facets of both conducting plates.

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2 pts d) Confirm that, on the lower surface of the upper plate, and also on the upper surface of the lower plate, σ_s and $\boldsymbol{J}_s$ satisfy the charge-current continuity equation.

Hint: $\boldsymbol{\nabla} \cdot \boldsymbol{V} = (\partial V_x/\partial x) + (\partial V_y/\partial y) + (\partial V_z/\partial z)$;

$$\boldsymbol{\nabla} \times \boldsymbol{V} = (\partial V_z/\partial y - \partial V_y/\partial z)\hat{\boldsymbol{x}} + (\partial V_x/\partial z - \partial V_z/\partial x)\hat{\boldsymbol{y}} + (\partial V_y/\partial x - \partial V_x/\partial y)\hat{\boldsymbol{z}}.$$

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4 pts **Problem 3**) a) In Newtonian physics, where $\boldsymbol{f} = m\boldsymbol{a} = m\mathrm{d}\boldsymbol{v}/\mathrm{d}t = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$, the time-rate of change of the kinetic energy $\mathcal{E} = ½mv^2$ of a point-particle of mass m equals the time-rate of work on the particle by the force $\boldsymbol{f}$; that is, $\mathrm{d}\mathcal{E}/\mathrm{d}t = \boldsymbol{f} \cdot \boldsymbol{v}$. Show that a similar relation holds in relativistic physics, where the total energy of the particle is $\mathcal{E} = \gamma mc^2$, while its linear momentum is $\boldsymbol{p} = \gamma m\boldsymbol{v}$, where $\gamma = 1/\sqrt{1-(v/c)^2}$ is the Lorentz factor.

2 pts b) In Newtonian physics, where the angular momentum of a point-particle with respect to the origin of coordinates is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, while the torque acting on the particle is $\boldsymbol{T} = \boldsymbol{r} \times \boldsymbol{f}$, the corresponding equation of motion is $\boldsymbol{T} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$. Show that the same equation of motion is applicable in relativistic dynamics as well. (It is assumed here that the point-particle, having rest-mass m and velocity $\boldsymbol{v}(t)$, is located at the point $\boldsymbol{r}(t)$ at time t.)

Hint: Considering that $\gamma = [1 - (\boldsymbol{v} \cdot \boldsymbol{v})/c^2]^{-½}$, the derivative of the Lorentz factor with respect to time is found to be

$$\mathrm{d}\gamma/\mathrm{d}t = -½\left(-\frac{2\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{v}/\mathrm{d}t}{c^2}\right)[1-(\boldsymbol{v}\cdot\boldsymbol{v})/c^2]^{-3/2} = (\gamma^3/c^2)\boldsymbol{v}\cdot(\mathrm{d}\boldsymbol{v}/\mathrm{d}t) = ½(\gamma^3/c^2)(\mathrm{d}v^2/\mathrm{d}t).$$
