

Problem 1) a) In analogy with Maxwell's 1st equation, the 4th equation must change to become

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = \rho_{\text{free}}^{(m)}(\mathbf{r}, t).$$

Similarly, in analogy with Maxwell's 2nd equation, the modified 3rd equation becomes

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mathbf{J}_{\text{free}}^{(m)}(\mathbf{r}, t) - \partial \mathbf{B}(\mathbf{r}, t) / \partial t.$$

The 1st and 2nd equations remain intact. To confirm the validity of the magnetic charge-current continuity equation, we take the divergence of the (modified) 3rd equation, invoke the fact that the divergence of curl always equals zero, and bring in the (modified) 4th equation, as follows:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{E}) &= -\nabla \cdot \mathbf{J}_{\text{free}}^{(m)} - \partial (\nabla \cdot \mathbf{B}) / \partial t = -\nabla \cdot \mathbf{J}_{\text{free}}^{(m)} - \partial \rho_{\text{free}}^{(m)} / \partial t \\ &\rightarrow \nabla \cdot \mathbf{J}_{\text{free}}^{(m)} + \partial \rho_{\text{free}}^{(m)} / \partial t = 0. \end{aligned} \quad \leftarrow \begin{array}{c} \text{magnetic charge-current} \\ \text{continuity equation} \end{array}$$

b) Upon dot-multiplying Maxwell's 2nd equation into $\mathbf{E}$ and the (modified) 3rd equation into $\mathbf{H}$, then subtracting one from the other, we find

$$\begin{aligned} \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}) &= \mathbf{E} \cdot \mathbf{J}_{\text{free}}^{(e)} + \mathbf{H} \cdot \mathbf{J}_{\text{free}}^{(m)} + \mathbf{E} \cdot (\partial \mathbf{P} / \partial t) + \mathbf{H} \cdot (\partial \mathbf{M} / \partial t) \\ &\quad \underbrace{-\nabla \cdot \mathbf{S}(\mathbf{r}, t)}_{\text{}} + \partial (1/2 \epsilon_0 \mathbf{E} \cdot \mathbf{E} + 1/2 \mu_0 \mathbf{H} \cdot \mathbf{H}) / \partial t. \end{aligned}$$

The new term appearing on the right-hand side of the above equation represents the time-rate of energy exchange between the $\mathbf{H}$ -field and the magnetic free current-density $\mathbf{J}_{\text{free}}^{(m)}$.

Problem 2) a) The specified EM fields satisfy all four of Maxwell's equations, as shown below.

$$\begin{aligned} \text{i)} \quad \nabla \cdot \mathbf{D} &= \partial(\epsilon_0 E_x) / \partial x = 0. \quad \leftarrow \rho_{\text{free}}(\mathbf{r}, t) = 0 \text{ in the gap region} \\ \text{iv)} \quad \nabla \cdot \mathbf{B} &= \partial(\mu_0 H_y) / \partial y = 0. \quad \leftarrow 1/(Z_0 c) = \sqrt{\mu_0 \epsilon_0} / \sqrt{\mu_0 / \epsilon_0} = \epsilon_0 \\ \text{ii)} \quad \nabla \times \mathbf{H} &= (\partial H_y / \partial x) \hat{\mathbf{z}} - (\partial H_y / \partial z) \hat{\mathbf{x}} = 0 + (E_0 / Z_0) (\omega / c) \sin[(\omega / c)z - \omega t] \hat{\mathbf{x}} \\ &= \epsilon_0 E_0 \omega \sin[(\omega / c)z - \omega t] \hat{\mathbf{x}}. \end{aligned}$$

$$\partial \mathbf{D} / \partial t = \epsilon_0 \frac{\partial}{\partial t} \{E_0 \cos[(\omega / c)z - \omega t] \hat{\mathbf{x}}\} = \epsilon_0 E_0 \omega \sin[(\omega / c)z - \omega t] \hat{\mathbf{x}}.$$

Considering that $\mathbf{J}_{\text{free}}(\mathbf{r}, t) = 0$ in the space between the plates, it is seen that Maxwell's 2nd equation, $\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t$, is satisfied.

$$\begin{aligned} \text{iii)} \quad \nabla \times \mathbf{E} &= (\partial E_x / \partial z) \hat{\mathbf{y}} - (\partial E_x / \partial y) \hat{\mathbf{z}} = -E_0 (\omega / c) \sin[(\omega / c)z - \omega t] \hat{\mathbf{y}} - 0 \\ &= -(E_0 \omega / c) \sin[(\omega / c)z - \omega t] \hat{\mathbf{y}}. \end{aligned}$$

$$\mu_0 / Z_0 = \mu_0 / \sqrt{\mu_0 / \epsilon_0} = \sqrt{\mu_0 \epsilon_0} = 1 / c$$

$$\partial \mathbf{B} / \partial t = \mu_0 \frac{\partial}{\partial t} \{(E_0 / Z_0) \hat{\mathbf{y}} \cos[(\omega / c)z - \omega t]\} = (E_0 \omega / c) \sin[(\omega / c)z - \omega t] \hat{\mathbf{y}}.$$

Thus, Maxwell's 3rd equation, $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$, is satisfied as well.

$$\text{b)} \quad \mathbf{S} = \mathbf{E} \times \mathbf{H} = (E_0^2 / Z_0) \cos^2[(\omega / c)z - \omega t] \hat{\mathbf{z}}.$$

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\epsilon_0 E_0^2 \cos^2[(\omega/c)z - \omega t] + \frac{1}{2}\mu_0 (E_0/Z_0)^2 \cos^2[(\omega/c)z - \omega t] \\ &= \epsilon_0 E_0^2 \cos^2[(\omega/c)z - \omega t].\end{aligned}$$

To verify the conservation of energy, we must show that the Poynting theorem is satisfied. Using the above expressions of $\mathbf{S}$ and $\mathcal{E}$, we find

$$\begin{aligned}\nabla \cdot \mathbf{S}(\mathbf{r}, t) &= \partial S_z / \partial z = -2(E_0^2/Z_0)(\omega/c) \sin[(\omega/c)z - \omega t] \cos[(\omega/c)z - \omega t], \\ \partial \mathcal{E}(\mathbf{r}, t) / \partial t &= 2\epsilon_0 E_0^2 \omega \sin[(\omega/c)z - \omega t] \cos[(\omega/c)z - \omega t].\end{aligned}$$

Given that $1/(Z_0 c) = \sqrt{\mu_0 \epsilon_0} / \sqrt{\mu_0 / \epsilon_0} = \epsilon_0$, the preceding equations confirm that $\nabla \cdot \mathbf{S} + \partial \mathcal{E} / \partial t = 0$.

c) The $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$ fields vanish inside the perfect conductors. At the lower facet of the upper plate, the perpendicular component of the $\mathbf{D}$ -field is discontinuous, with the amount of the discontinuity being $\mathbf{D}_\perp = \epsilon_0 \mathbf{E}(x = d, y, z, t)$. Therefore,

$$\sigma_s(x = d, y, z, t) = -\epsilon_0 E_0 \cos[(\omega/c)z - \omega t].$$

The tangential $\mathbf{H}$ -field is also discontinuous at the lower facet of the upper plate, with the amount of discontinuity being $\mathbf{H}_\parallel = \mathbf{H}(x = d, y, z, t)$. Consequently,

$$\mathbf{J}_s(x = d, y, z, t) = -(E_0/Z_0) \cos[(\omega/c)z - \omega t] \hat{\mathbf{z}}.$$

In similar fashion, the surface charge and current densities on the upper facet of the lower plate are given by

$$\sigma_s(x = 0, y, z, t) = \epsilon_0 E_0 \cos[(\omega/c)z - \omega t],$$

$$\mathbf{J}_s(x = 0, y, z, t) = (E_0/Z_0) \cos[(\omega/c)z - \omega t] \hat{\mathbf{z}}.$$

d) It is now easy to verify that the charge-current continuity equation, $\nabla \cdot \mathbf{J}_s + \partial \sigma_s / \partial t = 0$, is satisfied. At the lower facet of the top plate, we have

$$\begin{aligned}\nabla \cdot \mathbf{J}_s &= \partial J_{z,s} / \partial z = (E_0/Z_0)(\omega/c) \sin[(\omega/c)z - \omega t], \\ \partial \sigma_s / \partial t &= -\epsilon_0 E_0 \omega \sin[(\omega/c)z - \omega t],\end{aligned}$$

which obviously satisfy the continuity equation. The same relation holds at the top facet of the bottom plate, where both charge-density and current-density are the opposite of those at the lower facet of the upper plate.

Problem 3) a) In accordance with the special theory of relativity,

$$d\mathcal{E}/dt = d(\gamma mc^2)/dt = [\frac{1}{2}(\gamma^3/c^2)(dv^2/dt)]mc^2 = \frac{1}{2}m\gamma^3(dv^2/dt).$$

Also,

$$\begin{aligned}\mathbf{f} \cdot \mathbf{v} &= (d\mathbf{p}/dt) \cdot \mathbf{v} = [d(\gamma m \mathbf{v})/dt] \cdot \mathbf{v} = [(d\gamma/dt)m\mathbf{v} + \gamma m(d\mathbf{v}/dt)] \cdot \mathbf{v} \\ &= \frac{1}{2}(\gamma^3/c^2)(dv^2/dt)m v^2 + \gamma m \mathbf{v} \cdot (d\mathbf{v}/dt) = \frac{1}{2}\gamma m(dv^2/dt)[\gamma^2(v/c)^2 + 1] \\ &= \frac{1}{2}\gamma m(dv^2/dt) \left[\frac{(v/c)^2}{1 - (v/c)^2} + 1 \right] = \frac{1}{2}\gamma m(dv^2/dt) \left[\frac{1}{1 - (v/c)^2} \right] = \frac{1}{2}m\gamma^3(dv^2/dt).\end{aligned}$$

Consequently, $d\mathcal{E}/dt = \mathbf{f} \cdot \mathbf{v}$ is a valid identity in special relativity as well.

b) $d\mathbf{L}/dt = d(\mathbf{r} \times \mathbf{p})/dt = (d\mathbf{r}/dt) \times \mathbf{p} + \mathbf{r} \times d\mathbf{p}/dt = \mathbf{v}(t) \times \gamma m \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{f}(t) = \mathbf{T}(t).$