PhD Qualifying Exam, August 2025

Opti 501, Day 1

<u>System of units: SI (or MKSA)</u>

a) Write Maxwell's differential equations in their most complete form, including contributions from free-charge and free-current densities, as well as those from polarization and magnetization sources. Explain the meaning of each symbol that appears in these equations.

b) Derive the charge-current continuity equation directly from Maxwell's equations, and explain the meaning of this equation. Be brief but precise.

c) Define the bound-electric-charge and bound-electric-current densities. Use these entities to eliminate the $\boldsymbol{D}$ and $\boldsymbol{H}$ fields from Maxwell's equations. (In other words, rewrite Maxwell's equations with the help of bound-charge and bound-current densities in such a way that only the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields would appear in the equations.)

d) Show that the bound-charge and bound-current densities of part (c) satisfy their own charge-current continuity equation.

Solution to Day 1 Problem)

a)
$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=\rho_{\text{free}}(\boldsymbol{r},t),$$
$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t,$$
$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t,$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0.$$

In these equations, $\boldsymbol{r}=x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}$ is an arbitrary point in space, while t is an arbitrary instant in time. $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{D}$ is the displacement, and $\boldsymbol{B}$ is the magnetic induction. The fields are related to each other, to the permittivity and permeability of free space, ε_o and μ_o, and to polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$, as follows:

$$\boldsymbol{D}(\boldsymbol{r},t)=\varepsilon_o\boldsymbol{E}(\boldsymbol{r},t)+\boldsymbol{P}(\boldsymbol{r},t),$$
$$\boldsymbol{B}(\boldsymbol{r},t)=\mu_o\boldsymbol{H}(\boldsymbol{r},t)+\boldsymbol{M}(\boldsymbol{r},t).$$

The sources of the electromagnetic fields (namely, $\boldsymbol{E}$ and $\boldsymbol{H}$) are the free charge-density ρ_{free}, free current-density $\boldsymbol{J}_{\text{free}}$, polarization $\boldsymbol{P}$ (which is the density of electric dipole moments), and magnetization $\boldsymbol{M}$ (the density of magnetic dipole moments). The operator $\partial/\partial t$ represents partial differentiation with respect to time, $\boldsymbol{\nabla}\cdot$ is the divergence operator, and $\boldsymbol{\nabla}\times$ is the curl operator. The divergence of a vector field such as $\boldsymbol{D}(\boldsymbol{r},t)$, which turns out to be a scalar field, is defined as the integral of $\boldsymbol{D}(\boldsymbol{r},t)$ over a small, closed surface, normalized by the enclosed volume. The curl of a vector field such as $\boldsymbol{E}(\boldsymbol{r},t)$, which turns out to be another vector field, when projected onto the surface normal of a small surface element, yields the line integral of $\boldsymbol{E}(\boldsymbol{r},t)$ around the boundary of the surface element, normalized by the elemental surface area.

b) To derive the charge-current continuity equation from Maxwell's equations, apply the divergence operator to both sides of the second (Maxwell-Ampere) equation. The divergence of curl is always equal to zero and, therefore, the left-hand-side of the equation becomes $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H})=0$. The right-hand side, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}+\partial(\boldsymbol{\nabla}\cdot\boldsymbol{D})/\partial t$, thus becomes zero. Maxwell's first equation (Gauss's law) now allows one to replace $\boldsymbol{\nabla}\cdot\boldsymbol{D}$ with ρ_{free}, yielding the continuity equation as $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}+\partial\rho_{\text{free}}/\partial t=0.$ This equation informs that the integrated free current over any closed surface is precisely balanced by changes in the electrical charge contained within the closed surface. If there is a net outflow of current, the charge within the closed surface must be decreasing, and if there is a net inflow of current, the charge within must be increasing.

c) In the first of Maxwell's equations, we substitute $\boldsymbol{D}=\varepsilon_o\boldsymbol{E}+\boldsymbol{P}$ and obtain

$$\boldsymbol{\nabla}\cdot(\varepsilon_o\boldsymbol{E}+\boldsymbol{P})=\rho_{\text{free}}\quad\rightarrow\quad\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho_{\text{free}}-\boldsymbol{\nabla}\cdot\boldsymbol{P}\quad\rightarrow\quad\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho_{\text{free}}+\rho_{\text{bound}}^{(e)}.$$

The bound-charge density is thus seen to be $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t).$

In the second Maxwell equation, we multiply both sides by μ_o, then add $\boldsymbol{\nabla}\times\boldsymbol{M}$ to both sides, in order to replace $\boldsymbol{H}$ with $\boldsymbol{B}$ through the identity $\boldsymbol{B}=\mu_o\boldsymbol{H}+\boldsymbol{M}$. We also use $\boldsymbol{D}=\varepsilon_o\boldsymbol{E}+\boldsymbol{P}$ on the right-hand side of the equation to get rid of $\boldsymbol{D}$. We will have

$$\mu_o \boldsymbol{\nabla} \times \boldsymbol{H} + \boldsymbol{\nabla} \times \boldsymbol{M} = \mu_o \boldsymbol{J}_{\text{free}} + \mu_o \frac{\partial(\varepsilon_o \boldsymbol{E} + \boldsymbol{P})}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{M}$$

$$\rightarrow \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_o (\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t + \mu_o^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}) + \mu_o \varepsilon_o \partial \boldsymbol{E}/\partial t$$

$$\rightarrow \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_o (\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(e)}) + \mu_o \varepsilon_o \partial \boldsymbol{E}/\partial t.$$

The bound electric current-density is thus found to be $\boldsymbol{J}_{\text{bound}}^{(e)} = \partial \boldsymbol{P}/\partial t + \mu_o^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}$. Since the remaining Maxwell equations do not contain $\boldsymbol{D}$ and $\boldsymbol{H}$, they remain unchanged.

d) The divergence of $\boldsymbol{J}_{\text{bound}}^{(e)}$ is readily obtained, as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}^{(e)} = \partial(\boldsymbol{\nabla} \cdot \boldsymbol{P})/\partial t + \mu_o^{-1} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{M}).$$

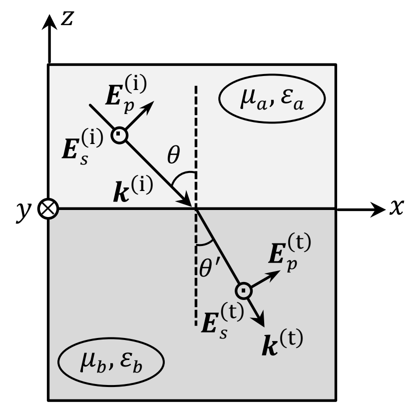
On the right-hand side of this equation, the divergence of the curl is always zero. Also the divergence of $\boldsymbol{P}(\boldsymbol{r}, t)$ is, by definition, $-\rho_{\text{bound}}^{(e)}$. Therefore, $\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}^{(e)} + \partial \rho_{\text{bound}}^{(e)}/\partial t = 0$. This is the charge-current continuity equation for the bound electrical charge and current defined in part (c).

PhD Qualifying Exam, August 2025

Opti 501, Day 2

<u>System of units: SI (or MKSA)</u>

Consider the flat interface between two linear, isotropic, homogeneous media specified by their relative permeability and permittivity at the incidence frequency, namely, (μ_a, ε_a) for the medium above, and (μ_b, ε_b) for the medium below the interface. These material parameters (i.e., $\mu_a, \mu_b, \varepsilon_a, \varepsilon_b$) are assumed to be real-valued and positive. A homogeneous plane-wave of frequency ω arrives at the interfacial xy-plane; the plane of incidence is xz, the incidence angle is θ, and the E-field components of the incident beam are $\boldsymbol{E}_p^{(\mathrm{i})}$ and $\boldsymbol{E}_s^{(\mathrm{i})}$, as indicated in the figure. The E-field components of the transmitted beam, also a homogeneous plane-wave, are $\boldsymbol{E}_p^{(\mathrm{t})}$ and $\boldsymbol{E}_s^{(\mathrm{t})}$, and the angle between the transmitted k-vector and the surface-normal is θ', as shown.



a) Invoking the dispersion relation $\boldsymbol{k} \cdot \boldsymbol{k} = (\omega/c)^2 \mu(\omega)\varepsilon(\omega)$, write expressions for the k-vectors of the incident and transmitted plane-waves shown in the above figure. ($c = 1/\sqrt{\mu_0 \varepsilon_0}$ is the speed of light in vacuum.)

b) Invoking Maxwell's boundary conditions, explain why the transmitted wave has the same frequency ω as the incident wave. What do these boundary conditions reveal about the relation between θ and θ'?

c) Use Maxwell's third equation, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, to determine both the incident and the transmitted H-field components. (As usual, $\boldsymbol{B} = \mu_0 \mu(\omega) \boldsymbol{H}$; you may use the impedance of free space, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$, to simplify the equations.)

d) Find the conditions under which the reflected beam for the p-polarized incident light vanishes.

e) Find the conditions under which the reflected beam for the s-polarized incident light vanishes.

Solution to Day 2 Problem)

a) $$|\boldsymbol{k}^{(\mathrm{i})}| = (\omega/c)\sqrt{\mu_a\varepsilon_a} \quad\rightarrow\quad \boldsymbol{k}^{(\mathrm{i})} = (\omega/c)\sqrt{\mu_a\varepsilon_a}(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}}). \tag{1}$$

$$|\boldsymbol{k}^{(\mathrm{t})}| = (\omega/c)\sqrt{\mu_b\varepsilon_b} \quad\rightarrow\quad \boldsymbol{k}^{(\mathrm{t})} = (\omega/c)\sqrt{\mu_b\varepsilon_b}(\sin\theta'\,\hat{\boldsymbol{x}} - \cos\theta'\,\hat{\boldsymbol{z}}). \tag{2}$$

b) Maxwell's boundary conditions require that $\boldsymbol{E}_\parallel, \boldsymbol{H}_\parallel, \boldsymbol{D}_\perp$, and $\boldsymbol{B}_\perp$ be continuous at the interface. Each field has a phase-factor $e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, which reduces to $e^{\mathrm{i}(k_x x + k_y y - \omega t)}$ when the interfacial plane is chosen to be the xy-plane at $z = 0$. Since the continuity conditions pertain to the fields immediately above and immediately below the interface at all times t, the frequencies of the incident, reflected, and transmitted beams must be identical. In particular, the frequency of the transmitted beam must be the same as the frequency ω of the incident beam.

Similarly, the continuity conditions are satisfied for all values of the coordinate y at the interfacial plane if and only if the k_y values of the incident, reflected, and transmitted beams are identical. Since our choice of xz as the plane of incidence automatically sets the k_y component of $\boldsymbol{k}^{(\mathrm{i})}$ to zero, we conclude that the k_y components of $\boldsymbol{k}^{(\mathrm{r})}$ and $\boldsymbol{k}^{(\mathrm{t})}$ must be zero as well.

Finally, the satisfaction of the boundary conditions for all values of the coordinate x at the interfacial plane requires that the k_x values of the incident, reflected, and transmitted beams be identical. In particular, setting $k_x^{(\mathrm{i})} = k_x^{(\mathrm{t})}$, we find from Eqs.(1) and (2) that the angles θ and θ' must be related as follows:

$$(\omega/c)\sqrt{\mu_a\varepsilon_a}\sin\theta = (\omega/c)\sqrt{\mu_b\varepsilon_b}\sin\theta' \quad\rightarrow\quad \sin\theta' = \sqrt{(\mu_a\varepsilon_a)/(\mu_b\varepsilon_b)}\sin\theta. \tag{3}$$

c) From $\boldsymbol{\nabla}\times\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = -(\partial/\partial t)\left[\mu_0\mu(\omega)\boldsymbol{H}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]$ we find $\boldsymbol{k}\times\boldsymbol{E}_0 = \mu_0\mu(\omega)\omega\boldsymbol{H}_0$, which leads to $(\omega/c)\sqrt{\mu(\omega)\varepsilon(\omega)}\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0 = \mu_0\mu(\omega)\omega\boldsymbol{H}_0$ and, therefore, $\boldsymbol{H}_0 = \sqrt{\varepsilon/\mu}\,\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0/Z_0$. For the incident plane-wave, this equation yields

$$\begin{aligned}
\boldsymbol{H}_0^{(\mathrm{i})} &= \sqrt{\varepsilon_a/\mu_a}\,\hat{\boldsymbol{\kappa}}^{(\mathrm{i})}\times\boldsymbol{E}_0^{(\mathrm{i})}/Z_0\\
&= Z_0^{-1}\sqrt{\varepsilon_a/\mu_a}\,(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})\times\left[E_p^{(\mathrm{i})}\cos\theta\,\hat{\boldsymbol{x}} + E_s^{(\mathrm{i})}\hat{\boldsymbol{y}} + E_p^{(\mathrm{i})}\sin\theta\,\hat{\boldsymbol{z}}\right]\\
&= Z_0^{-1}\sqrt{\varepsilon_a/\mu_a}\left[E_s^{(\mathrm{i})}\cos\theta\,\hat{\boldsymbol{x}} - E_p^{(\mathrm{i})}(\sin^2\theta + \cos^2\theta)\hat{\boldsymbol{y}} + E_s^{(\mathrm{i})}\sin\theta\,\hat{\boldsymbol{z}}\right]\\
&= Z_0^{-1}\sqrt{\varepsilon_a/\mu_a}\left[E_s^{(\mathrm{i})}\cos\theta\,\hat{\boldsymbol{x}} - E_p^{(\mathrm{i})}\hat{\boldsymbol{y}} + E_s^{(\mathrm{i})}\sin\theta\,\hat{\boldsymbol{z}}\right].
\end{aligned} \tag{4}$$

Similarly, for the transmitted plane-wave, we will have

$$\boldsymbol{H}_0^{(\mathrm{t})} = \sqrt{\varepsilon_b/\mu_b}\,\hat{\boldsymbol{\kappa}}^{(\mathrm{t})}\times\boldsymbol{E}_0^{(\mathrm{t})}/Z_0 = Z_0^{-1}\sqrt{\varepsilon_b/\mu_b}\left[E_s^{(\mathrm{t})}\cos\theta'\,\hat{\boldsymbol{x}} - E_p^{(\mathrm{t})}\hat{\boldsymbol{y}} + E_s^{(\mathrm{t})}\sin\theta'\,\hat{\boldsymbol{z}}\right]. \tag{5}$$

d) In the absence of a reflected beam, the continuity conditions for $\boldsymbol{E}_\parallel$ and $\boldsymbol{H}_\parallel$ of p-polarized light become

$$E_x^{(\mathrm{i})} = E_x^{(\mathrm{t})} \quad\rightarrow\quad E_p^{(\mathrm{i})}\cos\theta = E_p^{(\mathrm{t})}\cos\theta'. \tag{6}$$

$\boxed{\text{See Eqs.(4) and (5)}} \rightarrow$ $$H_y^{(\mathrm{i})} = H_y^{(\mathrm{t})} \quad\rightarrow\quad \sqrt{\varepsilon_a/\mu_a}\,E_p^{(\mathrm{i})} = \sqrt{\varepsilon_b/\mu_b}\,E_p^{(\mathrm{t})}. \tag{7}$$

Substituting for $E_p^{(\mathrm{t})}$ from Eq.(7) into Eq.(6), and recalling the relation between θ and θ' as given by Eq.(3), we find

$$\cancel{E_p^{(\mathrm{i})}}\sqrt{1-\sin^2\theta} = \sqrt{\mu_b\varepsilon_a/\mu_a\varepsilon_b}\,\cancel{E_p^{(\mathrm{i})}}\sqrt{1-\sin^2\theta'}$$

$$\rightarrow\;\; 1-\sin^2\theta = (\mu_b\varepsilon_a/\mu_a\varepsilon_b)[1-(\mu_a\varepsilon_a/\mu_b\varepsilon_b)\sin^2\theta] \;\;\rightarrow\;\; \sin\theta = \sqrt{\frac{1-(\mu_b\varepsilon_a/\mu_a\varepsilon_b)}{1-(\varepsilon_a/\varepsilon_b)^2}}. \qquad (8)$$

If $\mu_a=\mu_b$, we will have $\sin\theta=\sqrt{\varepsilon_b/(\varepsilon_a+\varepsilon_b)}$, which leads to $\cos\theta=\sqrt{\varepsilon_a/(\varepsilon_a+\varepsilon_b)}$ and $\tan\theta=\sqrt{\varepsilon_b/\varepsilon_a}$. But this may also be written as $\tan\theta=\sqrt{\mu_b\varepsilon_b/\mu_a\varepsilon_a}=n_b/n_a$, which is the well-known result associated with p-light incidence at Brewster's angle when $\mu_a=\mu_b$.

e) In the case of an s-polarized incident beam, the reflected beam vanishes when the following continuity conditions for $\boldsymbol{E}_\parallel$ and $\boldsymbol{H}_\parallel$ are satisfied:

$$E_y^{(\mathrm{i})} = E_y^{(\mathrm{t})} \;\;\rightarrow\;\; E_s^{(\mathrm{i})} = E_s^{(\mathrm{t})}. \qquad (9)$$

See Eqs.(4) and (5)

$$H_x^{(\mathrm{i})} = H_x^{(\mathrm{t})} \;\;\rightarrow\;\; \sqrt{\varepsilon_a/\mu_a}\,E_s^{(\mathrm{i})}\cos\theta = \sqrt{\varepsilon_b/\mu_b}\,E_s^{(\mathrm{t})}\cos\theta'. \qquad (10)$$

Substituting for $E_s^{(\mathrm{t})}$ from Eq.(9) into Eq.(10), and recalling the relation between θ and θ' as given by Eq.(3), we find

$$(\varepsilon_a/\mu_a)(1-\sin^2\theta) = (\varepsilon_b/\mu_b)(1-\sin^2\theta')$$

$$\rightarrow\;\; (\mu_b\varepsilon_a/\mu_a\varepsilon_b)(1-\sin^2\theta) = 1-(\mu_a\varepsilon_a/\mu_b\varepsilon_b)\sin^2\theta \;\;\rightarrow\;\; \sin\theta = \sqrt{\frac{\mu_b(\mu_a\varepsilon_b-\mu_b\varepsilon_a)}{(\mu_a^2-\mu_b^2)\varepsilon_a}}. \qquad (11)$$

At optical frequencies, ordinary materials have $\mu_a=\mu_b\cong 1$, which does not allow for the existence of a Brewster's angle for s-polarized light. However, whenever $\mu_a\neq\mu_b$, if Eq.(11) yields an acceptable value for the angle θ (i.e., an angle in the range of 0° to 90°), then such a Brewster angle for s-light would exist. If it so happens that $\varepsilon_a=\varepsilon_b$ while $\mu_a\neq\mu_b$, we will have $\sin\theta=\sqrt{\mu_b/(\mu_a+\mu_b)}$, which leads to $\cos\theta=\sqrt{\mu_a/(\mu_a+\mu_b)}$ and $\tan\theta=\sqrt{\mu_b/\mu_a}$. Once again, this may be written as $\tan\theta=\sqrt{\mu_b\varepsilon_b/\mu_a\varepsilon_a}=n_b/n_a$, as was the case for p-polarized light.