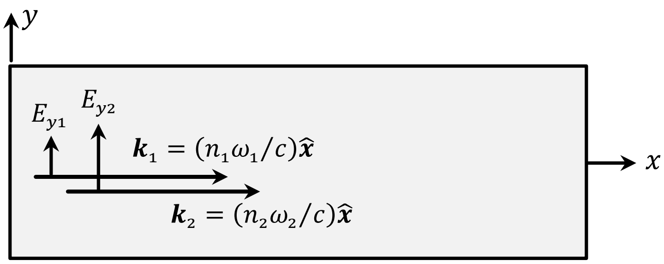
Opti 501 **Final Exam** (12/17/2024) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) Inside a transparent, linear, isotropic, and homogeneous medium of refractive index $n(\omega)$, two linearly-polarized plane-waves co-propagate along the x-axis. The E-field amplitudes and frequencies of the two waves are $(E_{y1}e^{\mathrm{i}\varphi_1}, \omega_1)$ and $(E_{y2}e^{\mathrm{i}\varphi_2}, \omega_2)$. The center frequency is $\omega_c = ½(\omega_1 + \omega_2)$ and the difference frequency is $\Delta\omega = \omega_2 - \omega_1$. Let the refractive indices at the two frequencies be $n_1 = n(\omega_1)$ and $n_2 = n(\omega_2)$. Define $E_s = ½(E_{y1} + E_{y2})$ and $E_d = ½(E_{y1} - E_{y2})$. Also, assuming that $\Delta\omega$ is sufficiently small, define the group refractive index n_g at the center frequency ω_c as $n_g(\omega_c) = (\omega_2 n_2 - \omega_1 n_1)/\Delta\omega \cong \mathrm{d}[\omega n(\omega)]/\mathrm{d}\omega|_{\omega=\omega_c}$.



4 pts a) Write an expression for the total (real-valued) electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ as a function of x and t.

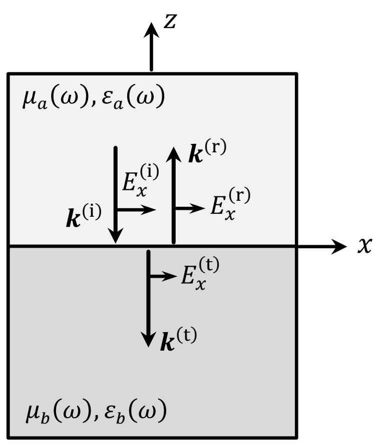
6 pts b) Combine the two E-fields to obtain a pair of beat signals with amplitudes $2E_s$ and $2E_d$.

3 pts c) Identify the envelopes and carriers of the two (co-propagating) beat signals.

Hint: $\cos a + \cos b = 2\cos[½(a+b)]\cos[½(a-b)]$ and $\cos a - \cos b = -2\sin[½(a+b)]\sin[½(a-b)]$.

Note that $E_{y1} = E_s + E_d$ and $E_{y2} = E_s - E_d$. You may also assume that $½(\omega_1 n_1 + \omega_2 n_2) \cong \omega_c n(\omega_c) = \omega_c n_c$.

Problem 2) A homogeneous, monochromatic, and linearly-polarized plane-wave of frequency ω arrives at normal incidence at the interface between a transparent medium a and an absorptive medium b. The optical parameters of the incidence and transmittance media are $\mu_a(\omega)$, $\varepsilon_a(\omega)$ and $\mu_b(\omega)$, $\varepsilon_b(\omega)$, respectively. You may assume that $\rho_{\text{free}}(\boldsymbol{r}, t) = 0$, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) = 0$, and $E_y^{(\mathrm{i})} = 0$.



2 pts a) Invoke the generalized Snell's law and the dispersion relation to identify $\boldsymbol{k}^{(\mathrm{i})}$, $\boldsymbol{k}^{(\mathrm{r})}$, and $\boldsymbol{k}^{(\mathrm{t})}$.

2 pts b) Use Maxwell's first equation, $\boldsymbol{\nabla} \cdot \boldsymbol{D}(\boldsymbol{r}, t) = \rho_{\text{free}}(\boldsymbol{r}, t)$, to show that $E_z^{(\mathrm{i})} = E_z^{(\mathrm{r})} = E_z^{(\mathrm{t})} = 0$.

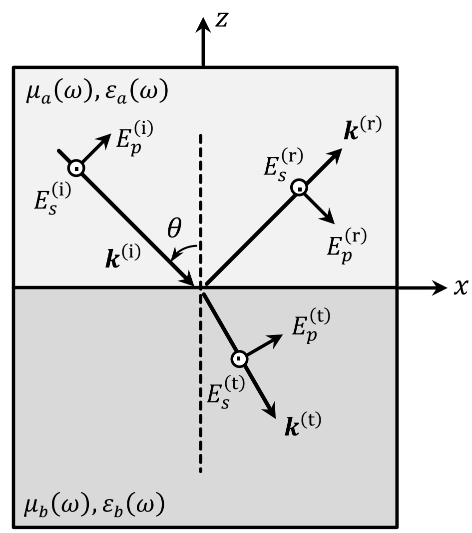
3 pts c) Use Maxwell's third equation, $\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\partial \boldsymbol{B}(\boldsymbol{r},t)/\partial t$, to identify the magnetic field amplitudes $\boldsymbol{H}_0^{(\mathrm{i})}$, $\boldsymbol{H}_0^{(\mathrm{r})}$, and $\boldsymbol{H}_0^{(\mathrm{t})}$.

3 pts d) Write the continuity conditions for the tangential $\boldsymbol{E}$ and $\boldsymbol{H}$ fields at the boundary at $z = 0$. Proceed to solve these equations to find the Fresnel reflection and transmission coefficients.

3 pts e) Write the complete expressions of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields for the reflected and transmitted beams.

1 pt f) Let $\varepsilon_b(\omega)$ be a complex number and $\mu_b(\omega)$ a large positive number. Examine the boundary conditions for E_x and H_y immediately above and immediately below the interface at $z = 0^{\pm}$. Explain how the presence of a surface-electric-current or a surface-magnetic-current might affect the continuity of the tangential H-field or that of the tangential E-field at the boundary.

Problem 3) A monochromatic plane-wave of frequency ω arrives at oblique incidence at the interface between a transparent medium a and an absorptive medium b. The optical parameters of the incidence and transmittance media are $\mu_a(\omega)$, $\varepsilon_a(\omega)$ and $\mu_b(\omega)$, $\varepsilon_b(\omega)$, respectively. The incident plane-wave is homogeneous, the plane of incidence is xz, the incidence angle is θ, and the incident E-field amplitudes ∥ and ⊥ to the plane of incidence are E_p and E_s, respectively.



3 pts a) Invoke the generalized Snell's law and the dispersion relation to relate $k_z^{(\mathrm{r})}$ and $k_z^{(\mathrm{t})}$ to $k_x^{(\mathrm{i})} = (\omega/c)n_a(\omega)\sin\theta$, to the frequency ω, and to the relevant material parameters.

3 pts b) Use Maxwell's first equation, $\boldsymbol{\nabla} \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho_{\mathrm{free}}(\boldsymbol{r},t)$, to relate the z-component of the E-field of each plane-wave to the corresponding k_x, k_z, E_x and E_y components. (**Hint**: Assume that $\rho_{\mathrm{free}} = 0$, and recall that $\boldsymbol{D} = \varepsilon_0\varepsilon(\omega)\boldsymbol{E}$, with $\varepsilon(\omega)$ incorporating the contributions of free as well as bound electrons.)

4 pts c) Invoke Maxwell's third equation, $\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\partial \boldsymbol{B}(\boldsymbol{r},t)/\partial t$, to express the H-field amplitudes of the incident, reflected, and transmitted beams in terms of the corresponding k-vectors and the E-field amplitudes (as well as other relevant parameters).

3 pts d) Write the boundary conditions pertaining to the continuity of the tangential $\boldsymbol{E}$ and $\boldsymbol{H}$ fields at the interfacial xy-plane at $z = 0$. Show that these four equations split into two independent sets of two equations each, one for the p-polarized light, the other for the s-polarized light.
