Opti 501 **2^{nd} Midterm Exam** (10/29/2024) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) The Fourier kernel in three-dimensional spacetime (i.e., two-dimensional space plus time) is $\boldsymbol{k}\cdot\boldsymbol{r}-\omega t = k_x x + k_y y - \omega t$.

1 pt a) Draw a diagram in the xy-plane that shows an arbitrary arrow for $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$, another arbitrary arrow for $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$, an angle φ between $\boldsymbol{k}$ and $\boldsymbol{r}$, and the location of all points in the xy-plane where, for the particular vector $\boldsymbol{k}$ that you have drawn, the dot-product $\boldsymbol{k}\cdot\boldsymbol{r}$ is constant.

1 pt b) In your diagram, identify the projection of $\boldsymbol{r}$ on $\boldsymbol{k}$, then explain why all the points in the xy-plane that have the same projection on $\boldsymbol{k}$ must have the same value for their dot-product $\boldsymbol{k}\cdot\boldsymbol{r}$.

1 pt c) Let the constant value of $\boldsymbol{k}\cdot\boldsymbol{r}$ be c_1. Pick a different constant c_2, somewhat greater than c_1, then show (on the same diagram) the location $\boldsymbol{r}$ of all the points in the xy-plane whose dot-product with your chosen vector $\boldsymbol{k}$ now equals c_2.

1 pt d) The distance between the two (straight and parallel) lines that you have identified in parts (a) and (c) is called a wavelength (denoted by λ) provided that $c_2 - c_1 = 2\pi$. What is the relation between λ and the magnitude k of the vector $\boldsymbol{k}$?

1 pt e) In the two-dimensional xy-space, a wavefront is a straight-line perpendicular to $\boldsymbol{k}$, whose distance from the origin is $d = r\cos\varphi$. The phase of the wavefront at time t is defined as $\Phi(\boldsymbol{r},t) = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t = kd - \omega t$. The wavefront moves along $\boldsymbol{k}$ with a velocity $\boldsymbol{V}$ in such a way that its phase at $d = Vt$ does *not* change with the passage of time. Show that $V = \omega/k$.

Problem 2) The scalar and vector potentials of classical electrodynamics are given in the Fourier domain as follows:

$$\psi(\boldsymbol{k},\omega) = \rho_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega)/\varepsilon_0[k^2 - (\omega/c)^2], \qquad \boldsymbol{A}(\boldsymbol{k},\omega) = \mu_0\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega)/[k^2 - (\omega/c)^2].$$

2 pts a) What are the relations between $\rho_{\text{total}}^{(\text{e})}$ and $\boldsymbol{J}_{\text{total}}^{(\text{e})}$ on the one hand, and the standard sources ρ_{free}, $\boldsymbol{J}_{\text{free}}$, $\boldsymbol{P}$, and $\boldsymbol{M}$ on the other? Express these relations first in the spacetime domain $(\boldsymbol{r},t)$, then translate them into the Fourier domain $(\boldsymbol{k},\omega)$.

1 pt b) Show that the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{total}}^{(\text{e})} + \partial\rho_{\text{total}}^{(\text{e})}/\partial t = 0$, is a direct consequence of Maxwell's 1st and 2nd equations, namely, $\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_{\text{total}}^{(\text{e})}$ and $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J}_{\text{total}}^{(\text{e})} + \mu_0\varepsilon_0\partial\boldsymbol{E}/\partial t$. **Hint**: $\boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\times\boldsymbol{V}(\boldsymbol{r},t)] = 0$ for any vector field $\boldsymbol{V}(\boldsymbol{r},t)$.

1 pt c) Express the above charge-current continuity equation in the Fourier domain $(\boldsymbol{k},\omega)$.

2 pts d) Show that the aforementioned $\psi(\boldsymbol{k},\omega)$ and $\boldsymbol{A}(\boldsymbol{k},\omega)$ satisfy the Lorenz gauge condition.

Problem 3) In the absence of all electromagnetic (EM) sources in free space, the total electric charge and current densities, namely, $\rho_{\text{total}}^{(\text{e})}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{r},t)$, will be zero. Nevertheless, EM fields and potentials can exist in free space if the identity $k^2 - (\omega/c)^2 = 0$ is satisfied. Under the circumstances, it is possible for a scalar potential plane-wave $\psi(\boldsymbol{r}, t) = \psi_0 e^{\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)}$ and the corresponding vector potential plane-wave $\boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{A}_0 e^{\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)}$ to reside in free space provided that $|\boldsymbol{k}_0| = \omega_0/c$.

1 pt a) What is the relation between $\boldsymbol{A}_0$ and ψ_0 if the above plane-wave potentials are specified in the Lorenz gauge? Simplify your result to show that $A_{0\parallel} = \psi_0/c$. (**Note**: $\parallel$ and $\perp$ are relative to $\boldsymbol{k}_0$.)

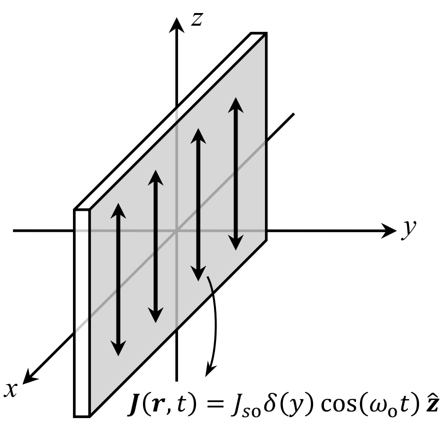
1 pt b) Find the amplitude $\boldsymbol{B}_0$ of the plane-wave's magnetic B-field in terms of $\boldsymbol{k}_0$ and $\boldsymbol{A}_{0\perp}$.

2 pts c) Find the amplitude $\boldsymbol{E}_0$ of the plane-wave's electric field in terms of ω_0, $\boldsymbol{k}_0$, $\boldsymbol{A}_0$ and ψ_0. Invoke the Lorenz gauge condition of part (a) to show that $\boldsymbol{E}_0 = \mathrm{i}\omega_0 \boldsymbol{A}_{0\perp}$.

2 pts d) Express the Poynting vector $\boldsymbol{S}(\boldsymbol{r}, t)$ of the plane-wave in terms of ω_0, $\boldsymbol{k}_0$, $\boldsymbol{A}_{0\perp}$ and the impedance Z_0 of free space. **Hint**: $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$; also $Z_0 = \mu_0 c$.

Problem 4) An infinitely large, thin, neutral (i.e., chargeless) sheet in the xz-plane carries the electric current-density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) = J_{s0}\delta(y)\cos(\omega_0 t)\,\hat{\boldsymbol{z}}$. Working in the Lorenz gauge, one can show that the scalar and vector potentials in the surrounding free space are given by

$$\psi(\boldsymbol{r}, t) = 0, \qquad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{Z_0 J_{s0}}{2\omega_0} \sin\left[\omega_0 \left(t - \frac{|y|}{c}\right)\right] \hat{\boldsymbol{z}}.$$



2 pts a) Find the radiated $\boldsymbol{E}$ and $\boldsymbol{H}$ fields on both sides of the sheet, i.e., in the free-space regions $y > 0$ and $y < 0$. **Hint**: $\boldsymbol{\nabla} \times \boldsymbol{V}(\boldsymbol{r}, t) = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{\boldsymbol{z}}.$

2 pts b) Confirm that your solutions for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields satisfy all four of Maxwell's boundary conditions at the surface of the sheet (i.e., in the xz-plane at $y = 0$).

2 pts c) Compute the Poynting vector $\boldsymbol{S}(\boldsymbol{r}, t)$ on both sides of the sheet, then determine the time-averaged rate (per unit area per unit time) at which the sheet radiates electromagnetic (EM) energy into its surrounding free space.

2 pts d) In the absence of polarization $\boldsymbol{P}(\boldsymbol{r}, t)$ and magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$, the rate of exchange of EM energy (per unit volume per unit time) between the radiated field and the material medium in this problem is $\boldsymbol{E}(\boldsymbol{r}, t) \cdot \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t)$. Show that the time-averaged radiated energy found in part (c) indeed equals the energy (per unit area per unit time) that is supplied by the mechanism that generates the sheet's current.
