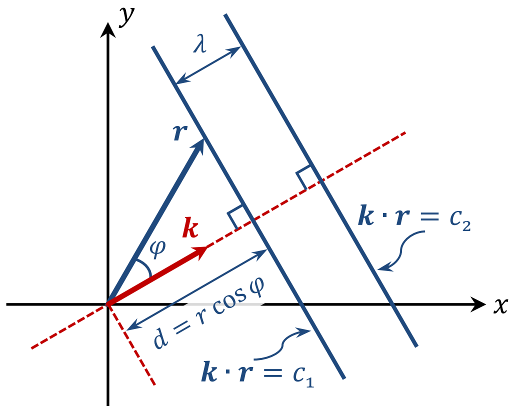
Opti 501 **2^{nd} Midterm Solutions** (10/29/2024) **Time: 75 minutes**

Problem 1) a) All points located on the straight-line drawn from $\boldsymbol{r}$ perpendicular to $\boldsymbol{k}$ (or $\perp$ to the extension of $\boldsymbol{k}$) have $\boldsymbol{k}\cdot\boldsymbol{r} = |\boldsymbol{k}||\boldsymbol{r}|\cos\varphi = kr\cos\varphi = kd$, where d is the distance from the origin of the coordinates to the foot of the perpendicular line thus drawn.



b) The projection of $\boldsymbol{r}$ on $\boldsymbol{k}$ has a length $d = r\cos\varphi$. Considering that all the points located on a straight-line drawn from $\boldsymbol{r}$ and $\perp$ to $\boldsymbol{k}$ have the same projection on $\boldsymbol{k}$, the dot-product $\boldsymbol{k}\cdot\boldsymbol{r}$ of the vector $\boldsymbol{k}$ with all the points $\boldsymbol{r}$ located on this straight-line equals $kr\cos\varphi = kd$, which is independent of the location of $\boldsymbol{r}$ on the perpendicular line.

c) Considering that $\boldsymbol{k}\cdot\boldsymbol{r} = kd = c_1$ and that k is a positive constant, if φ happens to be in the interval $[-90°, 90°]$, which is the case in the above diagram, then c_1 will be positive. In this case, choosing $c_2 > c_1$ results in a larger distance d from the origin and, therefore, a line $\perp$ to $\boldsymbol{k}$ that is parallel to and to the right of the previous straight-line (i.e., the one corresponding to $\boldsymbol{k}\cdot\boldsymbol{r} = c_1$). The new straight-line corresponding to c_2 is also shown in the above diagram.

d) Considering that $c_1 = kd_1$ and $c_2 = kd_2$, we have $k(d_2 - d_1) = c_2 - c_1 = 2\pi$. Denoting the distance $d_2 - d_1$ between the two (straight and parallel) lines by λ now yields $k = 2\pi/\lambda$.

e) Given that, for any given wavefront, $\Phi(\boldsymbol{r}, t) = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t = kd - \omega t$ does *not* change with the passage of time, we must have $\Delta\Phi = k\Delta d - \omega\Delta t = 0$. Consequently, $V = \Delta d/\Delta t = \omega/k$.

Problem 2) a) In the spacetime domain, we have

$$\rho_{\text{total}}^{(\text{e})}(\boldsymbol{r}, t) = \rho_{\text{free}}(\boldsymbol{r}, t) - \boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}, t),$$

$$\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{r}, t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) + \partial\boldsymbol{P}(\boldsymbol{r}, t)/\partial t + \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r}, t).$$

Translating these equations into the Fourier domain, we find

$$\rho_{\text{total}}^{(\text{e})}(\boldsymbol{k}, \omega) = \rho_{\text{free}}(\boldsymbol{k}, \omega) - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k}, \omega),$$

$$\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{k}, \omega) = \boldsymbol{J}_{\text{free}}(\boldsymbol{k}, \omega) - \mathrm{i}\omega\boldsymbol{P}(\boldsymbol{k}, \omega) + \mathrm{i}\mu_0^{-1}\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k}, \omega).$$

b) Applying the divergence operator to both sides of Maxwell's 2^{nd} equation, we arrive at

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = \mu_0\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{total}}^{(\text{e})} + \mu_0\varepsilon_0\partial(\boldsymbol{\nabla}\cdot\boldsymbol{E})/\partial t.$$

While the left-hand side of the above equation vanishes, the right-hand side further simplifies upon replacing $\varepsilon_0 \boldsymbol{\nabla}\cdot\boldsymbol{E}$ with $\rho_{\text{total}}^{(\text{e})}$ from Maxwell's 1st equation. This yields the charge-current continuity equation, as follows:

$$\cancel{\mu_0}\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{r},t) + \cancel{\mu_0}\partial\rho_{\text{total}}^{(\text{e})}(\boldsymbol{r},t)/\partial t = 0.$$

c) Translating the above charge-current continuity equation into the Fourier domain, we arrive at

$$\cancel{\mathrm{i}}\boldsymbol{k}\cdot\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega) - \cancel{\mathrm{i}}\omega\rho_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega) = 0.$$

d) The Lorenz gauge condition, $\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{r},t) + c^{-2}\partial\psi(\boldsymbol{r},t)/\partial t = 0$, translates to $\cancel{\mathrm{i}}\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},\omega) - \cancel{\mathrm{i}}(\omega/c^2)\psi(\boldsymbol{k},\omega) = 0$ in the Fourier domain. Substituting for $\boldsymbol{A}$ and ψ, we find

$$\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},\omega) - (\omega/c^2)\psi(\boldsymbol{k},\omega) = \frac{\mu_0\boldsymbol{k}\cdot\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega) - (\mu_0\varepsilon_0/\varepsilon_0)\omega\rho_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega)}{k^2 - (\omega/c)^2}.$$

The numerator of this equation is $\mu_0[\boldsymbol{k}\cdot\boldsymbol{J}_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega) - \omega\rho_{\text{total}}^{(\text{e})}(\boldsymbol{k},\omega)]$, which, according to the charge-current continuity equation, equals zero. The Lorenz gauge condition is thus satisfied.

Problem 3) a) Applying the Lorenz gauge identity, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{r},t) + c^{-2}\partial\psi(\boldsymbol{r},t)/\partial t = 0$, to the plane-wave scalar and vector potentials, we find that $\boldsymbol{k}_0\cdot\boldsymbol{A}_0 = (\omega_0/c^2)\psi_0$. This is equivalent to $k_0 A_{0\parallel} = (\omega_0/c^2)\psi_0$, where $A_{0\parallel}$ is the projection of $\boldsymbol{A}_0$ onto the k-vector $\boldsymbol{k}_0$. Given that $k_0 = \omega_0/c$, we arrive at $A_{0\parallel} = \psi_0/c$.

b) $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = \mathrm{i}\boldsymbol{k}_0\times\boldsymbol{A}_0 e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)} \rightarrow \boldsymbol{B}_0 = \mathrm{i}\boldsymbol{k}_0\times\boldsymbol{A}_0 = \mathrm{i}\boldsymbol{k}_0\times(\boldsymbol{A}_{0\parallel}+\boldsymbol{A}_{0\perp})$.

Recalling that $\boldsymbol{k}_0\times\boldsymbol{A}_{0\parallel} = (\omega_0/c)\hat{\boldsymbol{k}}_0\times A_{0\parallel}\hat{\boldsymbol{k}}_0 = 0$, we find $\boldsymbol{B}_0 = \mathrm{i}\boldsymbol{k}_0\times\boldsymbol{A}_{0\perp}$.

c) $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t = -\mathrm{i}\boldsymbol{k}_0\psi_0 e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)} + \mathrm{i}\omega_0\boldsymbol{A}_0 e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}$

$$\begin{aligned}\rightarrow\quad \boldsymbol{E}_0 &= \mathrm{i}\omega_0\boldsymbol{A}_0 - \mathrm{i}\boldsymbol{k}_0\psi_0 = \mathrm{i}\omega_0(\boldsymbol{A}_{0\parallel}+\boldsymbol{A}_{0\perp}) - \mathrm{i}\boldsymbol{k}_0\psi_0\\ &= \mathrm{i}\omega_0\boldsymbol{A}_{0\perp} + \mathrm{i}\omega_0(\psi_0/c)\hat{\boldsymbol{k}}_0 - \mathrm{i}(\omega_0/c)\hat{\boldsymbol{k}}_0\psi_0 \quad \leftarrow \boxed{\hat{\boldsymbol{k}}_0 = \boldsymbol{k}_0/k_0 \text{ is the unit-vector along } \boldsymbol{k}_0.}\\ &= \mathrm{i}\omega_0\boldsymbol{A}_{0\perp} + \mathrm{i}(\omega_0/c)(\cancel{\psi_0} - \cancel{\psi_0})\hat{\boldsymbol{k}}_0 = \mathrm{i}\omega_0\boldsymbol{A}_{0\perp}.\end{aligned}$$

d) To determine the Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$, we need the real parts of both $\boldsymbol{E}$ and $\boldsymbol{H}$ fields; that is

$$\begin{aligned}\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t)] &= \mathrm{Re}\big[\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}\big] = \mathrm{Re}\big[\mathrm{i}\omega_0\boldsymbol{A}_{0\perp}e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}\big]\\ &= \mathrm{Re}\big[\mathrm{i}\omega_0(\boldsymbol{A}'_{0\perp}+\mathrm{i}\boldsymbol{A}''_{0\perp})e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}\big] \quad \leftarrow \boxed{\boldsymbol{A}_{0\perp} = \boldsymbol{A}'_{0\perp}+\mathrm{i}\boldsymbol{A}''_{0\perp} \text{ is a complex vector.}}\\ &= -\omega_0[\boldsymbol{A}'_{0\perp}\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t) + \boldsymbol{A}''_{0\perp}\cos(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)].\end{aligned}$$

$$\begin{aligned}\mathrm{Re}[\boldsymbol{H}(\boldsymbol{r},t)] &= \mathrm{Re}\big[\mu_0^{-1}\boldsymbol{B}_0 e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}\big] = \mu_0^{-1}\boldsymbol{k}_0\times\mathrm{Re}\big[\mathrm{i}(\boldsymbol{A}'_{0\perp}+\mathrm{i}\boldsymbol{A}''_{0\perp})e^{\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}\big]\\ &= -\mu_0^{-1}\boldsymbol{k}_0\times[\boldsymbol{A}'_{0\perp}\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t) + \boldsymbol{A}''_{0\perp}\cos(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)].\end{aligned}$$

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{r},t) &= \mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t)]\times\mathrm{Re}[\boldsymbol{H}(\boldsymbol{r},t)] \quad \boxed{\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t}\\ &= \mu_0^{-1}\omega_0[\boldsymbol{A}'_{0\perp}\sin(\cdots) + \boldsymbol{A}''_{0\perp}\cos(\cdots)]\times\{\boldsymbol{k}_0\times[\boldsymbol{A}'_{0\perp}\sin(\cdots) + \boldsymbol{A}''_{0\perp}\cos(\cdots)]\}\\ &= \mu_0^{-1}\omega_0\{[\boldsymbol{A}'_{0\perp}\sin(\cdots) + \boldsymbol{A}''_{0\perp}\cos(\cdots)]\cdot[\boldsymbol{A}'_{0\perp}\sin(\cdots) + \boldsymbol{A}''_{0\perp}\cos(\cdots)]\}\boldsymbol{k}_0\end{aligned}$$

$$-\mu_0^{-1}\omega_0\{[(\boldsymbol{A}'_{0\perp}\cdot\boldsymbol{k}_0)^{\nearrow 0}\sin(\cdots)+(\boldsymbol{A}''_{0\perp}\cdot\boldsymbol{k}_0)^{\nearrow 0}\cos(\cdots)]\}[\boldsymbol{A}'_{0\perp}\sin(\cdots)+\boldsymbol{A}''_{0\perp}\cos(\cdots)]$$

$$=Z_0^{-1}\omega_0^2[A'^2_{0\perp}\sin^2(\cdots)+A''^2_{0\perp}\cos^2(\cdots)+2\boldsymbol{A}'_{0\perp}\cdot\boldsymbol{A}''_{0\perp}\sin(\cdots)\cos(\cdots)]\hat{\boldsymbol{k}}_0.$$

The time-averaged Poynting vector is found to be $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=(\omega_0^2/2Z_0)(A'^2_{0\perp}+A''^2_{0\perp})\hat{\boldsymbol{k}}_0$.

Problem 4) a) $\boldsymbol{E}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t=-\tfrac{1}{2}Z_0J_{s0}\cos[\omega_0(t-|y|/c)]\,\hat{\boldsymbol{z}}$.

$$\boldsymbol{H}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{B}=\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{A}=\mu_0^{-1}\frac{\partial A_z}{\partial y}\hat{\boldsymbol{x}}$$

$$=\mu_0^{-1}\left(\frac{Z_0J_{s0}}{2\omega_0}\right)\left(-\frac{\omega_0}{c}\right)\text{sign}(y)\cos[\omega_0(t-|y|/c)]\,\hat{\boldsymbol{x}}$$

$$=-\tfrac{1}{2}\,\text{sign}(y)J_{s0}\cos[\omega_0(t-|y|/c)]\,\hat{\boldsymbol{x}}$$

b) Maxwell's 1st boundary condition is satisfied since there are no surface charges (i.e., $\sigma_{s0}=0$), and since $D_\perp=D_y=\varepsilon_0E_y$ is zero on both sides of the sheet.

Maxwell's 2nd boundary condition is satisfied since, in the xz-plane of the sheet at $y=0$, the surface-current-density $\boldsymbol{J}_s$ is given by $J_{s0}\cos(\omega_0t)\,\hat{\boldsymbol{z}}$, while the parallel H-fields at the sheet's front and back surfaces at $y=0^\pm$ are $\boldsymbol{H}_\parallel=H_x\hat{\boldsymbol{x}}=\mp\tfrac{1}{2}J_{s0}\cos(\omega_0t)\,\hat{\boldsymbol{x}}$. The discontinuity of $\boldsymbol{H}_\parallel$ is thus seen to be equal to the magnitude J_s of $\boldsymbol{J}_s$, while the direction of $\boldsymbol{H}_\parallel$ is orthogonal to that of $\boldsymbol{J}_s$ and in compliance with the right-hand rule.

Maxwell's 3rd boundary condition at $y=0^\pm$ is satisfied since $\boldsymbol{E}_\parallel=-\tfrac{1}{2}Z_0J_{s0}\cos(\omega_0t)\,\hat{\boldsymbol{z}}$ is continuous at the sheet's surface. This E-field also exists within the sheet, exerting a Lorentz force on the sheet's surface-current-density $\boldsymbol{J}_s$.

Maxwell's 4th boundary condition at $y=0^\pm$ is satisfied since the perpendicular B-field, i.e., $B_y=\mu_0H_y=0$, satisfies the continuity condition at the front and back sides of the sheet.

c) $\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)=\tfrac{1}{4}\,\text{sign}(y)Z_0J_{s0}^2\cos^2[\omega_0(t-|y|/c)]\,\hat{\boldsymbol{y}}$.

The time-averaged rate of EM energy flow into the surrounding free space on either side of the sheet is $\tfrac{1}{8}Z_0J_{s0}^2$, considering that $\langle\cos^2(\omega_0t)\rangle=\tfrac{1}{2}$. Combining the energy flux on both sides, the total rate of energy outflow from the sheet turns out to be $2\langle S(\boldsymbol{r},t)\rangle=\tfrac{1}{4}Z_0J_{s0}^2$.

d) $\int_{z=-\infty}^{\infty}\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)\text{d}z=[-\tfrac{1}{2}Z_0J_{s0}\cos(\omega_0t)\,\hat{\boldsymbol{z}}]\cdot[J_{s0}\cos(\omega_0t)\,\hat{\boldsymbol{z}}]=-\tfrac{1}{2}Z_0J_{s0}^2\cos^2(\omega_0t)$.

The time-average of the expression on the right-hand side of the above equation is $-\tfrac{1}{4}Z_0J_{s0}^2$, which is equal in magnitude and opposite in sign to the time-averaged rate of energy outflow found in part (c). The minus sign in the expression of $\langle\boldsymbol{E}\cdot\boldsymbol{J}_{\text{free}}\rangle$ indicates that the radiated E-field *extracts* energy from the current sheet — as opposed to delivering energy to the current sheet.
