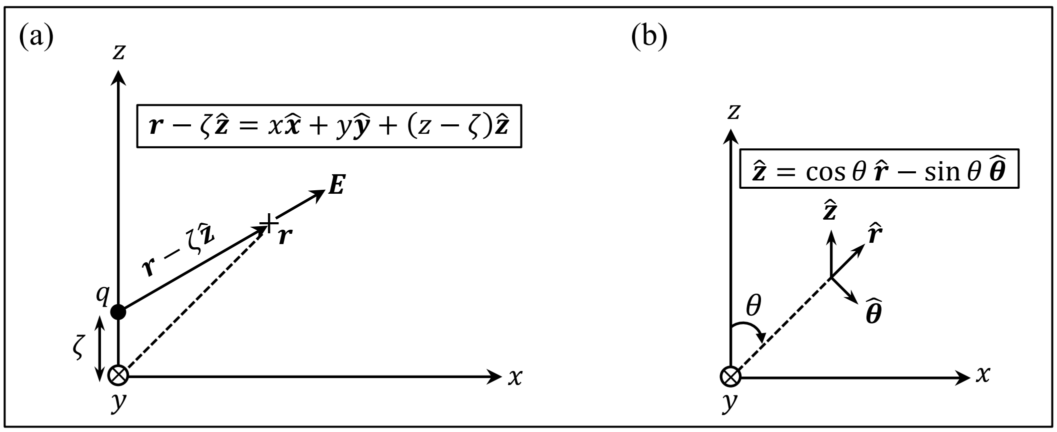
Opti 501 **1^{st} Midterm Exam** (9/26/2024) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) Consider a point-charge q located at $(x, y, z) = (0, 0, \zeta)$ in free space (i.e., vacuum), as shown in figure (a) below.

2 pts a) At first, assume that $\zeta = 0$; that is, the charge is located at the origin of the coordinate system. Invoke the integral form of Maxwell's first equation, $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}}$, along with symmetry considerations to show that the E-field at an arbitrary point $\boldsymbol{r}$ in the surrounding space is given by $\boldsymbol{E}(\boldsymbol{r}) = q\boldsymbol{r}/(4\pi\varepsilon_0 r^3)$. This, of course, is the well-known Coulomb E-field of the point-charge. (**Note**: $\boldsymbol{r} = r\hat{\boldsymbol{r}}$)



2 pts b) When $\zeta \neq 0$, the expression of $\boldsymbol{E}(\boldsymbol{r})$ must be modified to properly represent the new location of the point-charge. Figure (a) shows that the vector connecting the location of the point-charge to the observation point $\boldsymbol{r}$ is given by $\boldsymbol{r} - \zeta\hat{\boldsymbol{z}}$. Write an expression for $\boldsymbol{E}(\boldsymbol{r})$ in terms of the Cartesian coordinates (x, y, z) of the observation point $\boldsymbol{r}$ and the location ζ of the point-charge q on the z-axis.

2 pts c) Fixing the observation point at $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, find the derivative of $\boldsymbol{E}(\boldsymbol{r})$ with respect to the variable ζ, then evaluate this derivative at $\zeta = 0$. In other words, compute $\partial\boldsymbol{E}(\boldsymbol{r})/\partial\zeta|_{\zeta=0}$.

Hint: The derivative with respect to ζ of $[a + (b-\zeta)^2]^{-3/2}$ is $3(b-\zeta)[a + (b-\zeta)^2]^{-5/2}$.

2 pts d) Express the formula for $\partial\boldsymbol{E}(\boldsymbol{r})/\partial\zeta|_{\zeta=0}$ obtained in part (c) in the spherical coordinate system, where $r = (x^2 + y^2 + z^2)^{1/2}$, $z = r\cos\theta$, and, as shown in figure (b), $\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$.

2 pts e) The expressions obtained for $\partial\boldsymbol{E}(\boldsymbol{r})/\partial\zeta|_{\zeta=0}$ in Cartesian coordinates (part c) and spherical coordinates (part d) are related to the E-field distribution in the space surrounding an electric point-dipole $p_0\hat{\boldsymbol{z}}$, located at $(x, y, z) = (0, 0, 0)$. Here, $p_0 = q\Delta\zeta$ (for sufficiently small $\Delta\zeta$) is the magnitude of the point-dipole. Explain the way in which the expression of $\partial\boldsymbol{E}(\boldsymbol{r})/\partial\zeta|_{\zeta=0}$ in spherical coordinates (i.e., part d) leads to $\boldsymbol{E}_{\text{dipole}}(\boldsymbol{r}) = p_0(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/(4\pi\varepsilon_0 r^3)$.

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5 pts **Problem 2**) Within the international system of units (SI), describe the units (or dimensions) of the electric charge-density ρ, surface electric charge-density σ_s, electric current-density $\boldsymbol{J}$, surface electric current-density $\boldsymbol{J}_s$, electric field $\boldsymbol{E}$, displacement field $\boldsymbol{D}$, magnetic fields $\boldsymbol{B}$ and $\boldsymbol{H}$, the Poynting vector $\boldsymbol{S}$, and the electromagnetic momentum-density $\boldsymbol{S}/c^2$.

Invoke the various equations and identities of classical electrodynamics (in conjunction with those of the other areas of science and engineering) to express the units of each of the above entities in terms of the fundamental SI units, namely, meter (m), kilogram (kg), second (s), and ampere (A).

Hint: A point-particle's position as a function of time in space, $\boldsymbol{r}(t)$, its velocity, $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$, and its acceleration, $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$, have been used in classical physics to describe Newton's law of motion $\boldsymbol{f} = m\boldsymbol{a} = m\,\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$. Thus, the SI units of force (newton) are $kg \cdot m/s^2$. Similarly, Maxwell's equations, such as $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}}$, and the Lorentz force law, $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B})$, can be called upon to relate the units of various physical entities to those of the fundamental dimensions.

Problem 3) A magnetic point-dipole $\boldsymbol{m} = m_0\hat{\boldsymbol{z}}$ sits at the origin of a spherical coordinate system (r, θ, φ). In the surrounding free space, the dipole's magnetic field is given by

$$\boldsymbol{H}(\boldsymbol{r}) = m_0(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/(4\pi\mu_0 r^3).$$

3 pts a) Considering that $\boldsymbol{B}(\boldsymbol{r}) = \mu_0\boldsymbol{H}(\boldsymbol{r})$ at $r \neq 0$, verify that Maxwell's 4th equation, $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}) = 0$, is satisfied everywhere in the surrounding space.

2 pts b) The standard formula for the divergence operator in spherical coordinates has a singularity at the origin, namely, at $r = 0$. Apply the definition of divergence (i.e., integral of the vector field over the closed surface of a small volume, normalized by the volume) to verify that $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}) = 0$ at the origin as well. **Hint**: $\int_0^{\pi} \sin\theta\cos\theta\,\mathrm{d}\theta = 0$.

3 pts c) In accordance with Maxwell's 2nd equation, confirm that $\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}) = 0$ everywhere in the dipole's surrounding space.

2 pts d) The standard formula for the curl operator in spherical coordinates has a singularity at the origin, namely, at $r = 0$. Apply the definition of curl (i.e., integral of the vector field around three small, oriented loops, each normalized by the corresponding surface area) to verify that $\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}) = 0$ at the origin as well.

Hint: In spherical coordinates, the divergence and curl of the vector field $\boldsymbol{V}(\boldsymbol{r})$ are given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{V}(\boldsymbol{r}) = \frac{\partial(r^2 V_r)}{r^2 \partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta V_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi},$$

$$\boldsymbol{\nabla} \times \boldsymbol{V}(\boldsymbol{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta V_\varphi)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right]\hat{\boldsymbol{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial(rV_\varphi)}{r\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}.$$
