Opti 501 1st Midterm Solutions (9/26/2024) Time: 75 minutes

Problem 1) a) The E-field of the point-charge q must be along the straight line connecting the location of q to the observation point $\boldsymbol{r}$. Any deviation from this line would require a symmetry-breaking internal structure for the point-charge, which is experimentally known to be lacking. Such a deviation from the direction of the straight line would also result in a nonzero integral of the E-field around a circular path, thus violating Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=0$. (For more detail, see Sec.2.9 of the textbook, and also Problem 2 of the first midterm exam, Fall 2021.)

Drawing a sphere of radius r centered at the point charge q, then applying Maxwell's first equation $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$ in integral form, namely, $\oint_{\text{surface}}\boldsymbol{D}\cdot\mathrm{d}s=\int_{\text{volume}}\rho_{\text{free}}\mathrm{d}v=q$, yields

$$\boldsymbol{D}(\boldsymbol{r})=\varepsilon_0\boldsymbol{E}(\boldsymbol{r})=q\hat{\boldsymbol{r}}/(4\pi r^2).$$

Consequently, $\boldsymbol{E}(\boldsymbol{r})=q\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2)=q\boldsymbol{r}/(4\pi\varepsilon_0 r^3)$.

b) $$\boldsymbol{E}(\boldsymbol{r})=(q/4\pi\varepsilon_0)[x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+(z-\zeta)\hat{\boldsymbol{z}}][x^2+y^2+(z-\zeta)^2]^{-3/2}.$$

c) $$\mathrm{d}\boldsymbol{E}(\boldsymbol{r})/\mathrm{d}\zeta=(q/4\pi\varepsilon_0)\{-\hat{\boldsymbol{z}}[x^2+y^2+(z-\zeta)^2]^{-3/2}$$
$$-(3/2)(-2)(z-\zeta)[x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+(z-\zeta)\hat{\boldsymbol{z}}][x^2+y^2+(z-\zeta)^2]^{-5/2}\}.$$

Therefore,

$$\mathrm{d}\boldsymbol{E}(\boldsymbol{r})/\mathrm{d}\zeta|_{\zeta=0}=(q/4\pi\varepsilon_0)[-(\hat{\boldsymbol{z}}/r^3)+3z(r\hat{\boldsymbol{r}}/r^5)].$$

d) Substitution for z and $\hat{\boldsymbol{z}}$ in terms of the spherical coordinates r,θ and unit-vectors $\hat{\boldsymbol{r}},\hat{\boldsymbol{\theta}}$ yields

$$\mathrm{d}\boldsymbol{E}(\boldsymbol{r})/\mathrm{d}\zeta|_{\zeta=0}=(q/4\pi\varepsilon_0)\big[-(\cos\theta\hat{\boldsymbol{r}}-\sin\theta\,\hat{\boldsymbol{\theta}})/r^3+3r\cos\theta\,(\hat{\boldsymbol{r}}/r^4)\big]$$
$$=q(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})/(4\pi\varepsilon_0 r^3).$$

e) Taking $\Delta\zeta$ to be sufficiently small that both sides of the preceding equation can be multiplied by $\mathrm{d}\zeta=\Delta\zeta$, we now have the E-fields of two identical point-charges q located at $z=\pm\Delta\zeta/2$, subtracted from each other. This, of course, is just the meaning of the derivative of $\boldsymbol{E}(\boldsymbol{r})$ with respect to ζ, evaluated at $\zeta=0$; it is tantamount to adding the E-fields at the observation point $\boldsymbol{r}$ of a pair of point-charges $\pm q$ located at $z=\pm\Delta\zeta/2$. The pair of point-charges $\pm q$ thus separated by $\Delta\zeta$ at the origin of coordinates, constitute an electric dipole of magnitude $p_0=q\Delta\zeta$ located at the origin and aligned with the z-axis. The resulting dipole moment is, therefore, $\boldsymbol{p}=p_0\hat{\boldsymbol{z}}$, whose E-field is readily seen from the preceding equation to be

$$\boldsymbol{E}_{\text{dipole}}(\boldsymbol{r})=p_0(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})/(4\pi\varepsilon_0 r^3).$$

Problem 2) The SI unit of electrical current I is $ampere\ (A)$. Since $I=\Delta Q/\Delta t$, the unit of electrical charge Q, known as $coulomb\ (C)$, is $A\cdot s$.

The unit of the electric charge-density ρ is $coulomb/m^3=A\cdot s/m^3$.

The surface electric charge-density σ_s has the units of $coulomb/m^2=A\cdot s/m^2$.

The unit of the electric current-density $\boldsymbol{J}$ (i.e., current per unit cross-sectional area) is A/m^2.

The surface current-density $\boldsymbol{J}_s$ has the units of A/m.

The unit of the electric field $\boldsymbol{E}$ is $volt/m$, which, from the E-field part of the Lorentz force law, $\boldsymbol{f} = q\boldsymbol{E}$, equals $newton/coulomb$, or $kg \cdot m/(A \cdot s^3)$.

The displacement field $\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$ has the same dimension as the polarization-density $\boldsymbol{P}$ (i.e., electric dipole moment $q\boldsymbol{d}$ divided by volume), whose unit is $coulomb/m^2$. Also, from $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}}$, we find the unit of $\boldsymbol{D}$ to be $coulomb/m^2 = A \cdot s/m^2$.

The unit of the magnetic induction $\boldsymbol{B}$ is $weber/m^2$. From the B-field part of the Lorentz force law, $\boldsymbol{f} = q\boldsymbol{V} \times \boldsymbol{B}$, the unit of $\boldsymbol{B}$ is found to be $newton \cdot s/(coulomb \cdot m) = kg/(A \cdot s^2)$.

Invoking Maxwell's equation $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial\boldsymbol{D}/\partial t$, the unit of the magnetic field $\boldsymbol{H}$ is found to be $ampere/m = A/m$. This is because $\boldsymbol{\nabla} \times \boldsymbol{H}$ has the dimension of $\boldsymbol{H}$ divided by that of length (or distance in space).

The unit of the Poynting vector $\boldsymbol{S}$, which represents the time-rate of flow of EM energy per unit area, is $watt/m^2 = joule/(s \cdot m^2) = (newton \cdot m)/(s \cdot m^2) = kg/s^3$.

The electromagnetic (EM) momentum-density $\boldsymbol{S}/c^2$ has the unit of $\boldsymbol{S}$ (i.e., kg/s^3), divided by that of squared velocity (m^2/s^2). Thus, the unit of the EM momentum-density is $kg/(m^2 \cdot s)$. Needless to say, this is the unit of momentum ($kg \cdot m/s$) divided by the unit of volume (m^3).

Problem 3) a) In the absence of magnetization $\boldsymbol{M}$ in the surrounding space (i.e., $r \neq 0$), we have

$$\boldsymbol{B}(\boldsymbol{r}) = \mu_0\boldsymbol{H}(\boldsymbol{r}) = m_0(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/(4\pi r^3).$$

Consequently,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \left(\frac{m_0}{4\pi}\right)\left[\frac{\partial(2\cos\theta/r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin^2\theta/r^3)}{\partial\theta}\right] = \left(\frac{m_0}{4\pi}\right)\left(-\frac{2\cos\theta}{r^4} + \frac{2\sin\theta\cos\theta}{r^4\sin\theta}\right) = 0.$$

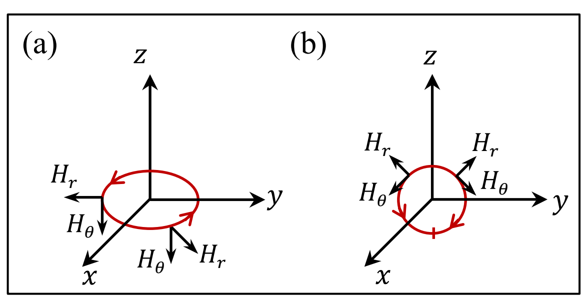
b) Take a small sphere of radius r around the origin. The B-field component that is perpendicular to the surface of the sphere is $B_r = m_0\cos\theta/(2\pi r^3)$. Integrating B_r over this spherical surface, we find

$$\oint_{\text{surface}} \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s} = \int_{\theta=0}^{\pi}\left(\frac{m_0\cos\theta}{2\pi r^3}\right)2\pi r^2\sin\theta\,\mathrm{d}\theta = (m_0/r)\int_{\theta=0}^{\pi}\sin\theta\cos\theta\,\mathrm{d}\theta$$
$$= (m_0/2r)\sin^2\theta\big|_{\theta=0}^{\pi} = 0.$$

c) The expression of the curl of $\boldsymbol{H}$ in spherical coordinates can be simplified since $H_\varphi = 0$, and also because H_r and H_θ do not depend on φ. Therefore,

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}} = \left(\frac{m_0}{4\pi\mu_0 r}\right)\left[\frac{\partial(\sin\theta/r^2)}{\partial r} - \frac{\partial(2\cos\theta/r^3)}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}$$
$$= \left(\frac{m_0}{4\pi\mu_0 r}\right)\left(-\frac{2\sin\theta}{r^3} + \frac{2\sin\theta}{r^3}\right)\hat{\boldsymbol{\varphi}} = 0.$$

d) Shown in Fig.(a) is a small circle of radius r within the xy-plane, centered at the origin of the coordinates. Neither the radial component H_r, nor the polar component H_θ, contribute to the loop integral. Therefore, the loop integral of $\boldsymbol{H}$ around this circle is zero, resulting in the z-component of $\boldsymbol{\nabla} \times \boldsymbol{H}$ at the origin being zero. Take a second small circle of radius r, again centered at the



origin, but this one perpendicular to the xy-plane (i.e., containing the z-axis), as depicted in Fig.(b). The azimuthal orientation of the circle is irrelevant here, due to the circular symmetry of $\boldsymbol{H}$ around the z-axis. Thus, the circle could be parallel to the xz-plane, or parallel to the yz-plane, etc. The radial component H_r of the H-field makes no contribution to the loop integral. The polar component H_θ contributes equally on the two semi-circles on either side of the z-axis; that is,

$$\pm \int_{\theta=0}^{\pi} H_\theta r \mathrm{d}\theta = \pm \left(\frac{m_0}{4\pi\mu_0 r^2}\right) \int_{\theta=0}^{\pi} \sin\theta \, \mathrm{d}\theta = \pm \left(\frac{m_0}{4\pi\mu_0 r^2}\right) (-\cos\theta)_{\theta=0}^{\pi} = \pm \frac{m_0}{2\pi\mu_0 r^2}.$$

The integrals over the two semi-circles are seen to be equal in magnitude and opposite in sign and, therefore, to cancel out. The end result is that the integral of $\boldsymbol{H}$ around any circular loop centered at the origin of coordinates and containing the z-axis is zero. All in all, we have now demonstrated that the curl of $\boldsymbol{H}$ evaluated at the origin of the coordinates is exactly equal to zero.