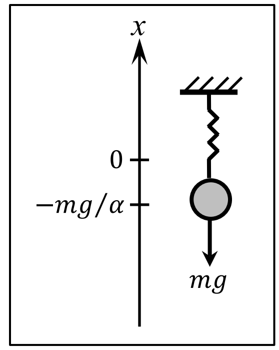
Temporal Evolution of a Harmonic Oscillator



Consider a harmonic oscillator consisting of a particle of mass m, subject to the gravitational force mg [units: $N = kg \cdot m/s^2$] and hanging from a spring whose real and positive spring constant is α [units: N/m]. The dynamic friction coefficient of the system is the positive real constant β [units: $N \cdot s/m$]. The equation of motion along the x-axis is given by Newton's second law, $\boldsymbol{f} = m\boldsymbol{a}$, as follows:

$$-mg\hat{\boldsymbol{x}} - \alpha x(t)\hat{\boldsymbol{x}} - \beta\dot{x}(t)\hat{\boldsymbol{x}} = m\ddot{x}(t)\hat{\boldsymbol{x}}$$

$$\rightarrow \qquad \ddot{x}(t) + (\beta/m)\dot{x}(t) + (\alpha/m)x(t) + g = 0. \tag{1}$$

The equilibrium position of the particle is readily seen from Eq.(1) to be $x = -mg/\alpha$. Let the initial position and velocity of the particle at $t = 0$ be specified as $x(0) = x_0$ and $\dot{x}(0) = 0$, respectively. Taking the temporal variations of $x(t)$ to be in the form of the exponential function $Ae^{\eta t}$ (with the parameters A and η as yet undetermined), we conjecture that

$$x(t) = -(mg/\alpha) + Ae^{\eta t}. \tag{2}$$

Substituting the above $x(t)$ into Eq.(1), we find

$$[\eta^2 + (\beta/m)\eta + (\alpha/m)]Ae^{\eta t} = 0. \tag{3}$$

The quadratic expression on the left-hand side of Eq.(3) is found to have the following two roots:

$$\eta^{\pm} = -(\beta/2m) \pm \sqrt{(\beta/2m)^2 - (\alpha/m)}. \tag{4}$$

Considering that the equation of motion, Eq.(1), is linear, both solutions η^+ and η^- can be accepted, with a linear combination of two exponential solutions (albeit with different coefficients) replacing the term $Ae^{\eta t}$ in Eq.(2). Thus, the general form of the solution of Eq.(1) is

$$x(t) = -(mg/\alpha) + Ae^{\eta^+ t} + Be^{\eta^- t}. \tag{5}$$

The coefficients A and B may now be found by enforcing the initial conditions; that is,

i) $\quad \dot{x}(t=0) = \left(A\eta^+ e^{\eta^+ t} + B\eta^- e^{\eta^- t}\right)\big|_{t=0} = 0 \quad \rightarrow \quad B = -A\eta^+/\eta^-. \qquad (6)$

ii) $\quad x(t=0) = -(mg/\alpha) + A + B = x_0 \quad \rightarrow \quad A - (A\eta^+/\eta^-) = x_0 + (mg/\alpha). \qquad (7)$

Consequently,

$$A = \frac{[x_0 + (mg/\alpha)]\eta^-}{\eta^- - \eta^+} = \frac{[x_0 + (mg/\alpha)](\sqrt{\beta^2 - 4m\alpha} + \beta)}{2\sqrt{\beta^2 - 4m\alpha}}, \qquad B = \frac{[x_0 + (mg/\alpha)](\sqrt{\beta^2 - 4m\alpha} - \beta)}{2\sqrt{\beta^2 - 4m\alpha}}. \tag{8}$$

Substitution into Eq.(5) yields

$$x(t) = -(mg/\alpha) + \tfrac{1}{2}[x_0 + (mg/\alpha)]e^{-\beta t/2m}$$
$$\times \left[\left(1 + \frac{\beta}{\sqrt{\beta^2 - 4m\alpha}}\right)e^{\sqrt{\beta^2 - 4m\alpha}\,t/2m} + \left(1 - \frac{\beta}{\sqrt{\beta^2 - 4m\alpha}}\right)e^{-\sqrt{\beta^2 - 4m\alpha}\,t/2m}\right]. \tag{9}$$

It is straightforward to verify that the above $x(t)$ satisfies $x(0) = x_0$ and $\dot{x}(0) = 0$. In the case of an over-damped system, where $\beta^2 > 4m\alpha$, Eq.(9) shows that, with increasing time, $x(t)$

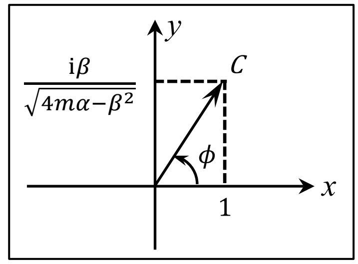
decays exponentially with two different time-constants $\tau^{\pm} = 2m/(\beta \mp \sqrt{\beta^2 - 4m\alpha})$.† In the case of an under-damped system, where $\beta^2 < 4m\alpha$, we will have $\sqrt{\beta^2 - 4m\alpha} = \mathrm{i}\sqrt{4m\alpha - \beta^2}$ and, therefore,

$$x(t) = -(mg/\alpha) + ½[x_0 + (mg/\alpha)]e^{-\beta t/2m}$$
$$\times\left[\left(1 - \frac{\mathrm{i}\beta}{\sqrt{4m\alpha-\beta^2}}\right)e^{\mathrm{i}\sqrt{4m\alpha-\beta^2}\,t/2m} + \left(1 + \frac{\mathrm{i}\beta}{\sqrt{4m\alpha-\beta^2}}\right)e^{-\mathrm{i}\sqrt{4m\alpha-\beta^2}\,t/2m}\right]. \quad (10)$$

Note that the two complex terms inside the square brackets on the right-hand side of Eq.(10) are conjugates and that, therefore, the overall solution $x(t)$ is real-valued. To simplify the above expression of $x(t)$, we introduce the complex constant C, as follows:

$$C = |C|e^{\mathrm{i}\phi} = 1 + \frac{\mathrm{i}\beta}{\sqrt{4m\alpha-\beta^2}}$$

$$\rightarrow\ |C| = \left(1 + \frac{\beta^2}{4m\alpha-\beta^2}\right)^{½} = \left(\frac{4m\alpha}{4m\alpha-\beta^2}\right)^{½},\ \ \phi = \tan^{-1}\left(\frac{\beta}{\sqrt{4m\alpha-\beta^2}}\right). \quad (11)$$



In streamlined form, Eq.(10) now becomes

$$x(t) = -(mg/\alpha) + ½[x_0 + (mg/\alpha)]e^{-\beta t/2m}$$
$$\times\left(|C|e^{-\mathrm{i}\phi}e^{\mathrm{i}\sqrt{4m\alpha-\beta^2}\,t/2m} + |C|e^{\mathrm{i}\phi}e^{-\mathrm{i}\sqrt{4m\alpha-\beta^2}\,t/2m}\right)$$

$$\rightarrow\ \ x(t) = -(mg/\alpha) + [x_0 + (mg/\alpha)]|C|e^{-\beta t/2m}\cos\left[\left(\sqrt{4m\alpha-\beta^2}/2m\right)t - \phi\right]. \quad (12)$$

As before, it is easy to verify that the above $x(t)$ satisfies the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. In this under-damped regime, the particle oscillates around its equilibrium position with frequency $\omega = \sqrt{(\alpha/m) - (\beta/2m)^2}$ and initial amplitude $[x_0 + (mg/\alpha)]|C|\cos\phi = x_0 + (mg/\alpha)$ at $t = 0$. The oscillation amplitude decays exponentially with a time-constant of $\tau = 2m/\beta$ as time progresses.

† For a critically-damped system, where $\beta^2 = 4m\alpha$, we will have $\eta^+ = \eta^- = -\beta/2m$. The homogeneous solutions of the equation of motion in this case will be $Ae^{-\beta t/2m}$ and $Bte^{-\beta t/2m}$. Enforcing the initial conditions then yields $A = x_0 + (mg/\alpha)$ and $B = (\beta/2m)A$. Consequently, $x(t) = -(mg/\alpha) + [x_0 + (mg/\alpha)][1 + (\beta/2m)t]e^{-\beta t/2m}$.