PhD Qualifying Exam, August 2024

Opti 501, Day 1

<u>System of units: SI (or MKSA)</u>

The electric field of a homogeneous plane-wave propagating in free space is given by $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$. Here, $\boldsymbol{E}_0 = \boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0''$, where $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are real-valued but otherwise arbitrary vectors, and the k-vector and the frequency ω are real-valued.

a) What is the dispersion relation in free space, and what does it have to say about the vector $\boldsymbol{k}$?

b) Invoke Maxwell's equation $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t)$ to show that $\boldsymbol{k}\cdot\boldsymbol{E}_0' = \boldsymbol{k}\cdot\boldsymbol{E}_0'' = 0$. (Recall that the plane-wave is in empty space, where no sources of the electromagnetic field reside.)

c) Invoke Maxwell's equation $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = -\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$ to find the magnetic H-field of the plane-wave.

d) Use the formula $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*)$ to evaluate the time-averaged Poynting vector. Explain (in words) the meaning and the significance of this time-averaged Poynting vector, specifically, its units and its relation to the electromagnetic energy of the plane-wave.

e) Write complete expressions for the real parts of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields. Proceed to write the complete expression for the (real-valued) Poynting vector $\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)$.

f) Verify that the time-averaged value of $\boldsymbol{S}(\boldsymbol{r},t)$ obtained in (e) agrees with the result of part (d).

Hint: $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$; $\cos^2\varphi = ½[1+\cos(2\varphi)]$; $\sin^2\varphi = ½[1-\cos(2\varphi)]$; $\sin(2\varphi) = 2\sin\varphi\cos\varphi$.

Solution to Problem 1) a) The dispersion relation, $k^2 = \boldsymbol{k}\cdot\boldsymbol{k} = (\omega/c)^2$, specifies the magnitude of the k-vector. In the present case, $\boldsymbol{k}$ is a real-valued vector having length ω/c and arbitrary orientation in three-dimensional xyz space. Introducing the unit-vector $\hat{\boldsymbol{\kappa}} = \boldsymbol{k}/k$, one writes $\boldsymbol{k} = (\omega/c)\hat{\boldsymbol{\kappa}}$.

b) In free space, where $\rho_{\text{free}}(\boldsymbol{r},t) = 0$ and $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t)$, Maxwell's first equation can be written as $\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = 0$. Applied to the plane-wave under consideration, this equation yields

$$\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0 \quad\rightarrow\quad \boldsymbol{k}\cdot(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'') = 0 + \mathrm{i}0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_0' = \boldsymbol{k}\cdot\boldsymbol{E}_0'' = 0.$$

c) Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = -\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$, applied to the plane-wave under consideration, yields $\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = \mathrm{i}\omega\mu_0\boldsymbol{H}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$. Solving for $\boldsymbol{H}_0$, we find $\boldsymbol{H}_0 = \boldsymbol{k}\times\boldsymbol{E}_0/(\mu_0\omega)$. Given that $\boldsymbol{k} = (\omega/c)\hat{\boldsymbol{\kappa}}$, where $\hat{\boldsymbol{\kappa}}$ is the unit-vector along the direction of $\boldsymbol{k}$, and taking into account the fact that $\mu_0 c = \mu_0/\sqrt{\mu_0\varepsilon_0} = \sqrt{\mu_0/\varepsilon_0} = Z_0$ (the impedance of free space), we find

$$\boldsymbol{H}(\boldsymbol{r},t) = Z_0^{-1}(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0' + \mathrm{i}\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0'')e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}.$$

d)
$$\begin{aligned}
\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle &= ½\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = ½\mathrm{Re}\left[\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\times\boldsymbol{H}_0^* e^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]\\
&= ½\mathrm{Re}[(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')\times Z_0^{-1}(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0' - \mathrm{i}\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0'')]\\
&= [\boldsymbol{E}_0'\times(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0') + \boldsymbol{E}_0''\times(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0'')]/(2Z_0)\\
&= [(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\hat{\boldsymbol{\kappa}} - (\cancelto{0}{\boldsymbol{E}_0'\cdot\hat{\boldsymbol{\kappa}}})\boldsymbol{E}_0' + (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}} - (\cancelto{0}{\boldsymbol{E}_0''\cdot\hat{\boldsymbol{\kappa}}})\boldsymbol{E}_0'']/(2Z_0)\\
&= (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}}/(2Z_0).
\end{aligned}$$

The time-averaged Poynting vector $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle$ is the rate of flow of electromagnetic energy per unit area per unit time. It is directed along the k-vector, and its SI units are Joule/(sec $\cdot$ m^2). The time-averaging is carried out over a single oscillation period $T = 2\pi/\omega$; that is,

$$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle = T^{-1}\int_{t_0}^{t_0+T}\boldsymbol{S}(\boldsymbol{r},t)\mathrm{d}t.$$

e)
$$\begin{aligned}
\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t)] &= \mathrm{Re}\left[\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right] = \mathrm{Re}\{(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + \mathrm{i}\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\\
&= \boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t).
\end{aligned}$$

Similarly, $\mathrm{Re}[\boldsymbol{H}(\boldsymbol{r},t)] = \boldsymbol{H}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{H}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)$. The real-valued Poynting vector may now be written as

$$\begin{aligned}
\boldsymbol{S}(\boldsymbol{r},t) &= [\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\times[\boldsymbol{H}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{H}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&= (\boldsymbol{E}_0'\times\boldsymbol{H}_0')\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + (\boldsymbol{E}_0''\times\boldsymbol{H}_0'')\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&\quad -(\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0')\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&= ½(\boldsymbol{E}_0'\times\boldsymbol{H}_0' + \boldsymbol{E}_0''\times\boldsymbol{H}_0'') + ½(\boldsymbol{E}_0'\times\boldsymbol{H}_0' - \boldsymbol{E}_0''\times\boldsymbol{H}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&\quad -½(\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0')\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].
\end{aligned}$$

The cross-products appearing in the above equation are readily evaluated, as follows:

$$\boldsymbol{E}_0' \times \boldsymbol{H}_0' = \boldsymbol{E}_0' \times Z_0^{-1}(\hat{\boldsymbol{\kappa}} \times \boldsymbol{E}_0') = Z_0^{-1}[(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0')\hat{\boldsymbol{\kappa}} - \overbrace{(\boldsymbol{E}_0' \cdot \hat{\boldsymbol{\kappa}})}^{0}\boldsymbol{E}_0'] = Z_0^{-1}(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0')\hat{\boldsymbol{\kappa}},$$

$$\boldsymbol{E}_0'' \times \boldsymbol{H}_0'' = Z_0^{-1}(\boldsymbol{E}_0'' \cdot \boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}}, \qquad \boldsymbol{E}_0' \times \boldsymbol{H}_0'' = Z_0^{-1}(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}}, \qquad \boldsymbol{E}_0'' \times \boldsymbol{H}_0' = Z_0^{-1}(\boldsymbol{E}_0'' \cdot \boldsymbol{E}_0')\hat{\boldsymbol{\kappa}}.$$

Substitution into the preceding expression of $\boldsymbol{S}(\boldsymbol{r},t)$ now yields

$$\boldsymbol{S}(\boldsymbol{r},t) = \tfrac{1}{2}Z_0^{-1}(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0' + \boldsymbol{E}_0'' \cdot \boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}} + \tfrac{1}{2}Z_0^{-1}(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0' - \boldsymbol{E}_0'' \cdot \boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}}\cos[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$
$$- Z_0^{-1}(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}}\sin[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)].$$

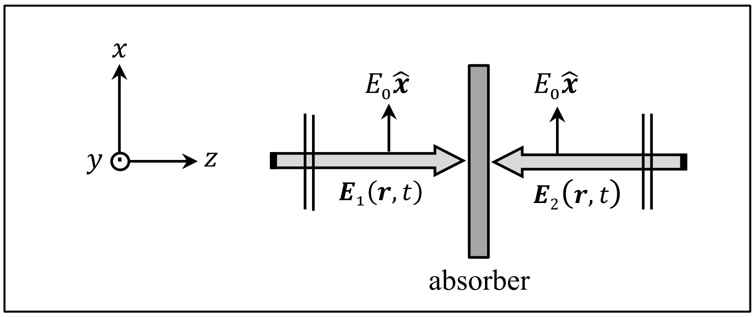
f) Considering that the time-averages of $\cos[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ and $\sin[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ vanish everywhere in space, the equation obtained in part (e) yields $\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle = \tfrac{1}{2}Z_0^{-1}(\boldsymbol{E}_0' \cdot \boldsymbol{E}_0' + \boldsymbol{E}_0'' \cdot \boldsymbol{E}_0'')\hat{\boldsymbol{\kappa}}$, in agreement with the result obtained in part (d).

PhD Qualifying Exam, August 2024

Opti 501, Day 2

<u>System of units: SI (or MKSA)</u>

The figure shows two counter-propagating plane-waves in free space arriving at a thin sheet of absorbing material located in the xy-plane at $z = 0$. In the half-space $z < 0$, the E-field of the right-propagating plane-wave is $\boldsymbol{E}_1(\boldsymbol{r}, t) = E_0\hat{\boldsymbol{x}}\cos[(\omega/c)(z - ct)]$, whereas in the half-space $z > 0$, the E-field of the left-propagating plane-wave is $\boldsymbol{E}_2(\boldsymbol{r}, t) = E_0\hat{\boldsymbol{x}}\cos[(\omega/c)(z + ct)]$.



a) Identify the oscillation frequency, the k-vector, and the polarization state of each plane-wave.

b) Find the magnetic fields $\boldsymbol{H}_1(\boldsymbol{r}, t)$ and $\boldsymbol{H}_2(\boldsymbol{r}, t)$ of both plane-waves, each in its respective half-space.

c) Find the Poynting vectors $\boldsymbol{S}_1(\boldsymbol{r}, t)$ and $\boldsymbol{S}_2(\boldsymbol{r}, t)$ of both plane-waves, each in its respective half-space.

d) Use the formula $\langle \boldsymbol{S}(\boldsymbol{r}, t)\rangle = ½\text{Re}(\boldsymbol{E} \times \boldsymbol{H}^*)$ to calculate the *time-averaged* Poynting vector for each plane-wave. Confirm that the results thus obtained agree with the results of part (c).

Note: In part (d), $\boldsymbol{E}$ and $\boldsymbol{H}$ refer to the complex version of the fields.

e) Invoke Maxwell's boundary conditions at the front and back facets of the thin-sheet absorber (located at $z = 0^+$ and $z = 0^-$) to determine the parallel component $\boldsymbol{E}_\parallel$ of the E-field as well as the perpendicular components $\boldsymbol{D}_\perp$ and $\boldsymbol{B}_\perp$ of the $\boldsymbol{D}$ and $\boldsymbol{B}$ fields inside the absorber.

f) Invoke Maxwell's boundary condition for the parallel component $\boldsymbol{H}_\parallel$ of the H-field to determine the electric current-density $\boldsymbol{J}_s$ inside the absorber.

Note: Since the absorber is assumed to be infinitesimally thin, its current-density can be treated as a surface-current-density, having the units of ampere/meter.

g) Confirm that the rate $\boldsymbol{E} \cdot \boldsymbol{J}_s$ at which electromagnetic energy is taken up by the absorber equals the rate of delivery of electromagnetic energy by the incoming pair of plane-waves.

Solution to Problem 2) a) The oscillation frequency is ω, the k-vectors are $\pm(\omega/c)\hat{\boldsymbol{z}}$, and the state of polarization of both plane-waves is linear along the x-axis.

b) From Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, we find $\mathrm{i}\boldsymbol{k}\times E_0\hat{\boldsymbol{x}} = \mathrm{i}\omega\mu_0\boldsymbol{H}_0$. Consequently, $\pm(\omega/c)\hat{\boldsymbol{z}}\times E_0\hat{\boldsymbol{x}} = \omega\mu_0\boldsymbol{H}_0$, which yields $\boldsymbol{H}_0 = \pm(E_0/Z_0)\hat{\boldsymbol{y}}$. As always, $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the speed of light in vacuum, while $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space. Consequently,

$$\boldsymbol{H}_{1,2}(\boldsymbol{r},t) = \pm(E_0/Z_0)\hat{\boldsymbol{y}}\cos[(\omega/c)(z\mp ct)].$$

c) $\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) = \pm(E_0^2/Z_0)\cos^2[(\omega/c)(z\mp ct)]\,\hat{\boldsymbol{z}}$. (The upper sign corresponds to the beam in the half-space $z<0$, while the lower sign represents the beam in $z>0$.)

d) $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}\big[E_0\hat{\boldsymbol{x}}e^{\mathrm{i}(\pm\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\times(\pm E_0/Z_0)\hat{\boldsymbol{y}}e^{-\mathrm{i}(\pm\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\big] = \pm½(E_0^2/Z_0)\hat{\boldsymbol{z}}$.

This same result is obtained by time-averaging $\boldsymbol{S}(\boldsymbol{r},t)$ obtained in part (c), considering that the time-average of the cosine-squared factor is $\langle\cos^2[(\omega/c)(z\mp ct)]\rangle = ½$.

e) At $z=0^+$ and 0^-, both E-fields are $E_0\hat{\boldsymbol{x}}\cos(\omega t)$; the continuity of $\boldsymbol{E}_\parallel$ extends this to become the E-field inside the thin-sheet absorber. There is no $\boldsymbol{D}_\perp$ or $\boldsymbol{B}_\perp$ in the free-space region outside the absorber, nor does the absorber carry any surface charges; therefore, both of these fields are zero inside the absorber.

f) The discontinuity of $\boldsymbol{H}_\parallel$ between $z=0^+$ and $z=0^-$ in accordance with the result obtained in part (b) yields

$$\boldsymbol{J}_s = 2(E_0/Z_0)\hat{\boldsymbol{x}}\cos(\omega t).$$

Note that $\boldsymbol{J}_s$ has the units of [ampere/meter], in agreement with the right-hand side of the above equation. Denoting the volume current-density inside the absorber by $\boldsymbol{J}$ [ampere/m^2], one can write $\boldsymbol{J}_s = \boldsymbol{J}d$, where d is the infinitesimally small thickness of the absorber.

g) The rate of electromagnetic energy uptake by the current-density $\boldsymbol{J}_s$ acted upon by the E-field within the absorber is $\boldsymbol{E}\cdot\boldsymbol{J}_s = 2(E_0^2/Z_0)\cos^2(\omega t)$, whose time-average is E_0^2/Z_0. This agrees with the time-averaged influx of energy in the two incoming plane-waves, as found in (d).

Digression. According to Problem (7.65) of the textbook, for a slab of thickness d and refractive index n surrounded by free space, the Fresnel reflection and transmission coefficients at normal incidence are

$$\rho = E_0^{(\mathrm{r})}/E_0^{(\mathrm{i})} = \frac{\rho_{12}+\rho_{23}\exp(\mathrm{i}4\pi nd/\lambda_0)}{1+\rho_{12}\rho_{23}\exp(\mathrm{i}4\pi nd/\lambda_0)}, \qquad \tau = E_0^{(\mathrm{t})}/E_0^{(\mathrm{i})} = \frac{\tau_{12}\tau_{23}\exp(\mathrm{i}2\pi nd/\lambda_0)}{1+\rho_{12}\rho_{23}\exp(\mathrm{i}4\pi nd/\lambda_0)},$$

where $\rho_{12} = (1-n)/(1+n)$, $\rho_{23} = (n-1)/(n+1)$, $\tau_{12} = 2/(1+n)$, $\tau_{23} = 2n/(n+1)$, and $\lambda_0 = 2\pi c/\omega$. For the absorber to be perfect, the reflection from one side must cancel out the transmission from the opposite side; that is,

$$\rho+\tau = 0 \quad\rightarrow\quad \rho_{12}+\rho_{23}\exp(\mathrm{i}4\pi nd/\lambda_0)+\tau_{12}\tau_{23}\exp(\mathrm{i}2\pi nd/\lambda_0) = 0$$

$$\rightarrow\quad \left(\frac{1-n}{1+n}\right)+\left(\frac{n-1}{n+1}\right)e^{\mathrm{i}4\pi nd/\lambda_0}+\frac{4n}{(n+1)^2}e^{\mathrm{i}2\pi nd/\lambda_0} = 0$$

$$\rightarrow \qquad e^{\mathrm{i}4\pi nd/\lambda_0} + \left(\frac{4n}{n^2-1}\right) e^{\mathrm{i}2\pi nd/\lambda_0} - 1 = 0.$$

The two solutions of the above quadratic equation are readily found to be

$$e^{\mathrm{i}2\pi nd/\lambda_0} = -\left(\frac{2n}{n^2-1}\right) \pm \left[\frac{4n^2}{(n^2-1)^2} + 1\right]^{½} = -\left(\frac{2n}{n^2-1}\right) \pm \left(\frac{n^2+1}{n^2-1}\right) = \pm \frac{(n \mp 1)^2}{(n-1)(n+1)}.$$

Of these, the second solution (lower sign) is unacceptable, as it causes the denominator in the expressions of ρ and τ to vanish. Consequently, the only acceptable solution is $e^{\mathrm{i}2\pi nd/\lambda_0} = (n-1)/(n+1)$. When the sheet thickness d is negligibly small, the refractive index n must be sufficiently large, thus permitting the following approximation:

$$e^{\mathrm{i}2\pi nd/\lambda_0} = \frac{n-1}{n+1} = \frac{1-(1/n)}{1+(1/n)} \cong \frac{e^{-1/n}}{e^{1/n}} = e^{-2/n}.$$

$\boxed{n^2(\omega) = \varepsilon(\omega) = 1 + \chi(\omega)}$

This leads to: $\mathrm{i}2\pi nd/\lambda_0 \cong -2/n \quad \rightarrow \quad \pi n^2 d/\lambda_0 \cong \mathrm{i} \quad \rightarrow \quad \pi[1+\chi(\omega)]d/\lambda_0 \cong \mathrm{i}$. In the limit when $d \rightarrow 0$ and $\chi(\omega) \rightarrow \infty$, the condition for perfect absorption becomes $\pi\chi(\omega)d/\lambda_0 = \mathrm{i}$. Now, in complex notation, the current-density $\boldsymbol{J}_s$ within the absorber is related to the absorber thickness d and the material polarization $\boldsymbol{P} = \varepsilon_0\chi(\omega)\boldsymbol{E}$ in the following way:

$$\boldsymbol{J}_s = (\partial\boldsymbol{P}/\partial t)d = -\mathrm{i}\omega\varepsilon_0\chi(\omega)E_0\hat{\boldsymbol{x}}d \underset{\boxed{\text{see part (f)}}}{=} 2(E_0/Z_0)\hat{\boldsymbol{x}} \;\rightarrow\; \varepsilon_0 Z_0\omega\chi(\omega)d = 2\mathrm{i} \;\rightarrow\; \pi\chi(\omega)d/\lambda_0 = \mathrm{i}.$$

The last equation agrees with the requirement for perfect absorption in the limit when $d \rightarrow 0$.

One may be wondering why the current sheet in this problem does *not* radiate a pair of electromagnetic plane-waves toward infinity, as was the case in the example described in Section **4**.10 of the textbook. The answer is that the radiated beams are hiding in plain sight! When the incoming plane-waves pass through the absorber sheet, their continuation on the opposite side get cancelled out by the radiated pair of plane-waves.