Solution to Problem 1)

a) $$\boldsymbol{k} = (\omega/c)\hat{\boldsymbol{\kappa}}.$$

b) $$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}} \;\rightarrow\; \boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E} + \cancelto{0}{\boldsymbol{P}}) = \cancelto{0}{\rho_{\text{free}}} \;\rightarrow\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = 0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0$$

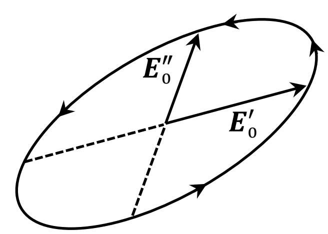
$$\rightarrow\; \mathrm{i}(\omega/c)\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0 \;\rightarrow\; \hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0 = 0 \;\rightarrow\; \hat{\boldsymbol{\kappa}}\cdot(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'') = 0$$

$$\rightarrow\; (\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0') + \mathrm{i}(\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0'') = 0 \;\rightarrow\; \hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0' = 0 \text{ and } \hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0'' = 0.$$

c) $$Re\left[(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right] = Re\{(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + \mathrm{i}\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}$$

$$= \boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t).$$

At any given point $\boldsymbol{r} = \boldsymbol{r}_0$, the E-field is a function of time. When $\sin(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t) = 0$, we will have $\cos(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t) = \pm 1$, in which case the field has its maximum amplitude along $\boldsymbol{E}_0'$. And when $\cos(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t) = 0$, we have $\sin(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t) = \pm 1$, in which case the field has its maximum amplitude along $\boldsymbol{E}_0''$. During each cycle of oscillation, the tip of the E-field vector traces an elliptical trajectory, as depicted in the figure. The plane-wave is linearly polarized when either $\boldsymbol{E}_0' = 0$ or $\boldsymbol{E}_0'' = 0$, or when $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are parallel (or anti-parallel) to each other. The plane-wave is circularly polarized when $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are perpendicular to each other and have equal magnitudes. Considering that the tip of the E-field vector travels from $\boldsymbol{E}_0'$ toward $\boldsymbol{E}_0''$, the plane-wave will be right or left circularly polarized depending on the relative orientation of these two vectors.



d) $$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = \mathrm{i}\omega\mu_0\boldsymbol{H}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

$$\rightarrow\; \mu_0\omega\boldsymbol{H}_0 = (\omega/c)\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0 \;\rightarrow\; \boldsymbol{H}_0 = \hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0/\mu_0 c \;\rightarrow\; \boldsymbol{H}_0 = \hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0/Z_0.$$

e) $$\mathcal{E}_E(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_0|Re[\boldsymbol{E}(\boldsymbol{r},t)]|^2 = \tfrac{1}{2}\varepsilon_0|\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)|^2$$

$$= \tfrac{1}{2}\varepsilon_0\{\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0''\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0'\cdot\boldsymbol{E}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}$$

$$= \tfrac{1}{4}\varepsilon_0\{(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') + (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' - \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] - 2\boldsymbol{E}_0'\cdot\boldsymbol{E}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

Upon time-averaging, the oscillatory terms of the above expression vanish, yielding $\langle\mathcal{E}_E(\boldsymbol{r},t)\rangle = \tfrac{1}{4}\varepsilon_0(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')$, which can equivalently be written as $\langle\mathcal{E}_E(\boldsymbol{r},t)\rangle = \tfrac{1}{4}\varepsilon_0\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*$. A similar procedure applied to the H-field yields

$$\langle\mathcal{E}_H(\boldsymbol{r},t)\rangle = \tfrac{1}{2}\mu_0\langle|Re[\boldsymbol{H}(\boldsymbol{r},t)]|^2\rangle = \tfrac{1}{4}\mu_0\boldsymbol{H}_0\cdot\boldsymbol{H}_0^* = \tfrac{1}{4}(\mu_0/Z_0^2)(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0)\cdot(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0^*)$$

$$= \tfrac{1}{4}\varepsilon_0\hat{\boldsymbol{\kappa}}\cdot[\boldsymbol{E}_0\times(\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0^*)] = \tfrac{1}{4}\varepsilon_0\hat{\boldsymbol{\kappa}}\cdot[(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)\hat{\boldsymbol{\kappa}} - \cancelto{0}{(\boldsymbol{E}_0\cdot\hat{\boldsymbol{\kappa}})}\boldsymbol{E}_0^*] = \tfrac{1}{4}\varepsilon_0\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*.$$

The E-field and H-field energy densities are thus seen to be identical. As for the time-averaged Poynting vector, we will have

$$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle = \tfrac{1}{2}Re[\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}^*(\boldsymbol{r},t)] = \tfrac{1}{2}Re\left[\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\times\boldsymbol{H}_0^* e^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]$$

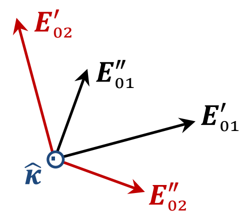
$$= \tfrac{1}{2}Re[\boldsymbol{E}_0 \times (\hat{\boldsymbol{\kappa}} \times \boldsymbol{E}_0^*/Z_0)] = (2Z_0)^{-1}Re[(\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*)\hat{\boldsymbol{\kappa}} - \cancelto{0}{(\boldsymbol{E}_0 \cdot \hat{\boldsymbol{\kappa}})}\boldsymbol{E}_0^*]$$
$$= (2Z_0)^{-1}(\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*)\hat{\boldsymbol{\kappa}}.$$

Note that the magnitude of the time-averaged Poynting vector equals the sum of the E-field and H-field energy densities, namely, $\tfrac{1}{2}\varepsilon_0\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*$, multiplied by the speed c of light in vacuum. This is the sense in which the Poynting vector yields the rate of flow of electromagnetic energy per unit area per unit time.

f) The time-averaged energy densities and the Poynting vector of a plane-wave are seen to be proportional to $\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*$. Thus, the relevant entity for the superposition $(\alpha\boldsymbol{E}_{01} + \beta\boldsymbol{E}_{02})e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$ is

$$(\alpha\boldsymbol{E}_{01} + \beta\boldsymbol{E}_{02}) \cdot (\alpha\boldsymbol{E}_{01} + \beta\boldsymbol{E}_{02})^* = |\alpha|^2\boldsymbol{E}_{01} \cdot \boldsymbol{E}_{01}^* + |\beta|^2\boldsymbol{E}_{02} \cdot \boldsymbol{E}_{02}^* + 2Re(\alpha\beta^*\boldsymbol{E}_{01} \cdot \boldsymbol{E}_{02}^*).$$

For the energy densities and the Poynting vector of the superposed plane-wave to be linear combinations of the corresponding entities for the constituent beams (for all values of α and β), it is necessary as well as sufficient to have $\boldsymbol{E}_{01} \cdot \boldsymbol{E}_{02}^* = 0$. This is equivalent to requiring that $\boldsymbol{E}_{01}' \cdot \boldsymbol{E}_{02}' + \boldsymbol{E}_{01}'' \cdot \boldsymbol{E}_{02}'' = 0$ and also $\boldsymbol{E}_{01}' \cdot \boldsymbol{E}_{02}'' - \boldsymbol{E}_{01}'' \cdot \boldsymbol{E}_{02}' = 0$. One way to achieve this, as the figure suggests, is by rotating $\boldsymbol{E}_{01}'$ around $\hat{\boldsymbol{\kappa}}$ by 90°, say, counterclockwise, to arrive at $\boldsymbol{E}_{02}'$, then rotating $\boldsymbol{E}_{01}''$ around $\hat{\boldsymbol{\kappa}}$ by 90°, this time clockwise, to arrive at $\boldsymbol{E}_{02}''$. In this way, the orthogonality constraint $\boldsymbol{E}_{01} \cdot \boldsymbol{E}_{02}^* = 0$ is satisfied and the two polarization states $\boldsymbol{E}_{01}$ and $\boldsymbol{E}_{02}$ of the $(\omega, \hat{\boldsymbol{\kappa}})$ plane-wave become mutually orthogonal.



Solution to Problem 2)

a) $$|\boldsymbol{k}^{(\text{i})}| = (\omega/c)\sqrt{\mu_a\varepsilon_a} \quad \rightarrow \quad \boldsymbol{k}^{(\text{i})} = (\omega/c)\sqrt{\mu_a\varepsilon_a}(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}}). \tag{1}$$

$$|\boldsymbol{k}^{(\text{t})}| = (\omega/c)\sqrt{\mu_b\varepsilon_b} \quad \rightarrow \quad \boldsymbol{k}^{(\text{t})} = (\omega/c)\sqrt{\mu_b\varepsilon_b}(\sin\theta'\,\hat{\boldsymbol{x}} - \cos\theta'\,\hat{\boldsymbol{z}}). \tag{2}$$

b) Maxwell's boundary conditions require that $\boldsymbol{E}_{\|}, \boldsymbol{H}_{\|}, \boldsymbol{D}_{\perp}$, and $\boldsymbol{B}_{\perp}$ be continuous at the interface. Each field has a phase-factor $e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$, which reduces to $e^{\mathrm{i}(k_x x + k_y y - \omega t)}$ when the interfacial plane is chosen to be the xy-plane at $z = 0$. Since the continuity conditions pertain to the fields immediately above and immediately below the interface at all times t, the frequencies of the incident, reflected, and transmitted beams must be identical. In particular, the frequency of the transmitted beam must be the same as the frequency ω of the incident beam.

Similarly, the continuity conditions are satisfied for all values of the coordinate y at the interfacial plane if and only if the k_y values of the incident, reflected, and transmitted beams are identical. Since our choice of xz as the plane of incidence automatically sets the k_y component of $\boldsymbol{k}^{(\text{i})}$ to zero, we conclude that the k_y components of $\boldsymbol{k}^{(\text{r})}$ and $\boldsymbol{k}^{(\text{t})}$ must be zero as well.

Finally, the satisfaction of the boundary conditions for all values of the coordinate x at the interfacial plane requires that the k_x values of the incident, reflected, and transmitted beams be identical. In particular, setting $k_x^{(\text{i})} = k_x^{(\text{t})}$, we find from Eqs.(1) and (2) that the angles θ and θ' must be related as follows:

$$(\omega/c)\sqrt{\mu_a\varepsilon_a}\sin\theta = (\omega/c)\sqrt{\mu_b\varepsilon_b}\sin\theta' \quad \rightarrow \quad \sin\theta' = \sqrt{(\mu_a\varepsilon_a)/(\mu_b\varepsilon_b)}\sin\theta. \tag{3}$$

c) From $\boldsymbol{\nabla}\times\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} = -(\partial/\partial t)\left[\mu_0\mu(\omega)\boldsymbol{H}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\right]$ we find $\boldsymbol{k}\times\boldsymbol{E}_0 = \mu_0\mu(\omega)\omega\boldsymbol{H}_0$, which leads to $(\omega/c)\sqrt{\mu(\omega)\varepsilon(\omega)}\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0 = \mu_0\mu(\omega)\omega\boldsymbol{H}_0$ and, therefore, $\boldsymbol{H}_0 = \sqrt{\varepsilon/\mu}\,\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0/Z_0$. For the incident plane-wave, this equation yields

$$\begin{aligned}
\boldsymbol{H}_0^{(\text{i})} &= \sqrt{\varepsilon_a/\mu_a}\,\hat{\boldsymbol{\kappa}}^{(\text{i})}\times\boldsymbol{E}_0^{(\text{i})}/Z_0 \\
&= Z_0^{-1}\sqrt{\varepsilon_a/\mu_a}\,(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})\times\left[E_p^{(\text{i})}\cos\theta\,\hat{\boldsymbol{x}} + E_s^{(\text{i})}\hat{\boldsymbol{y}} + E_p^{(\text{i})}\sin\theta\,\hat{\boldsymbol{z}}\right] \\
&= Z_0^{-1}\sqrt{\varepsilon_a/\mu_a}\left[E_s^{(\text{i})}\cos\theta\,\hat{\boldsymbol{x}} - E_p^{(\text{i})}(\sin^2\theta + \cos^2\theta)\hat{\boldsymbol{y}} + E_s^{(\text{i})}\sin\theta\,\hat{\boldsymbol{z}}\right] \\
&= Z_0^{-1}\sqrt{\varepsilon_a/\mu_a}\left[E_s^{(\text{i})}\cos\theta\,\hat{\boldsymbol{x}} - E_p^{(\text{i})}\hat{\boldsymbol{y}} + E_s^{(\text{i})}\sin\theta\,\hat{\boldsymbol{z}}\right].
\end{aligned} \tag{4}$$

Similarly, for the transmitted plane-wave, we will have

$$\boldsymbol{H}_0^{(\text{t})} = \sqrt{\varepsilon_b/\mu_b}\,\hat{\boldsymbol{\kappa}}^{(\text{t})}\times\boldsymbol{E}_0^{(\text{t})}/Z_0 = Z_0^{-1}\sqrt{\varepsilon_b/\mu_b}\left[E_s^{(\text{t})}\cos\theta'\,\hat{\boldsymbol{x}} - E_p^{(\text{t})}\hat{\boldsymbol{y}} + E_s^{(\text{t})}\sin\theta'\,\hat{\boldsymbol{z}}\right]. \tag{5}$$

d) In the absence of a reflected beam, the continuity conditions for $\boldsymbol{E}_{\|}$ and $\boldsymbol{H}_{\|}$ of p-polarized light become

$$E_x^{(\text{i})} = E_x^{(\text{t})} \quad \rightarrow \quad E_p^{(\text{i})}\cos\theta = E_p^{(\text{t})}\cos\theta'. \tag{6}$$

See Eqs.(4) and (5) →
$$H_y^{(\text{i})} = H_y^{(\text{t})} \quad \rightarrow \quad \sqrt{\varepsilon_a/\mu_a}\,E_p^{(\text{i})} = \sqrt{\varepsilon_b/\mu_b}\,E_p^{(\text{t})}. \tag{7}$$

Substituting for $E_p^{(\text{t})}$ from Eq.(7) into Eq.(6), and recalling the relation between θ and θ' as given by Eq.(3), we find

$$\cancel{E_p^{(\mathrm{i})}}\sqrt{1-\sin^2\theta} = \sqrt{\mu_b\varepsilon_a/\mu_a\varepsilon_b}\,\cancel{E_p^{(\mathrm{i})}}\sqrt{1-\sin^2\theta'}$$

$$\rightarrow\ 1-\sin^2\theta = (\mu_b\varepsilon_a/\mu_a\varepsilon_b)[1-(\mu_a\varepsilon_a/\mu_b\varepsilon_b)\sin^2\theta] \quad\rightarrow\quad \sin\theta = \sqrt{\frac{1-(\mu_b\varepsilon_a/\mu_a\varepsilon_b)}{1-(\varepsilon_a/\varepsilon_b)^2}}. \qquad (8)$$

If $\mu_a = \mu_b$, we will have $\sin\theta = \sqrt{\varepsilon_b/(\varepsilon_a+\varepsilon_b)}$, which leads to $\cos\theta = \sqrt{\varepsilon_a/(\varepsilon_a+\varepsilon_b)}$ and $\tan\theta = \sqrt{\varepsilon_b/\varepsilon_a}$. But this may also be written as $\tan\theta = \sqrt{\mu_b\varepsilon_b/\mu_a\varepsilon_a} = n_b/n_a$, which is the well-known result associated with p-light incidence at Brewster's angle when $\mu_a = \mu_b$.

e) In the case of an s-polarized incident beam, the reflected beam vanishes when the following continuity conditions for $\boldsymbol{E}_\parallel$ and $\boldsymbol{H}_\parallel$ are satisfied:

$$E_y^{(\mathrm{i})} = E_y^{(\mathrm{t})} \quad\rightarrow\quad E_s^{(\mathrm{i})} = E_s^{(\mathrm{t})}. \qquad (9)$$

See Eqs.(4) and (5) →

$$H_x^{(\mathrm{i})} = H_x^{(\mathrm{t})} \quad\rightarrow\quad \sqrt{\varepsilon_a/\mu_a}\,E_s^{(\mathrm{i})}\cos\theta = \sqrt{\varepsilon_b/\mu_b}\,E_s^{(\mathrm{t})}\cos\theta'. \qquad (10)$$

Substituting for $E_s^{(\mathrm{t})}$ from Eq.(9) into Eq.(10), and recalling the relation between θ and θ' as given by Eq.(3), we find

$$(\varepsilon_a/\mu_a)(1-\sin^2\theta) = (\varepsilon_b/\mu_b)(1-\sin^2\theta')$$

$$\rightarrow\ (\mu_b\varepsilon_a/\mu_a\varepsilon_b)(1-\sin^2\theta) = 1-(\mu_a\varepsilon_a/\mu_b\varepsilon_b)\sin^2\theta \quad\rightarrow\quad \sin\theta = \sqrt{\frac{\mu_b(\mu_a\varepsilon_b-\mu_b\varepsilon_a)}{(\mu_a^2-\mu_b^2)\varepsilon_a}}. \qquad (11)$$

At optical frequencies, ordinary materials have $\mu_a = \mu_b \cong 1$, which does not allow for the existence of a Brewster's angle for s-polarized light. However, whenever $\mu_a \neq \mu_b$, if Eq.(11) yields an acceptable value for the angle θ (i.e., an angle in the range of 0° to 90°), then such a Brewster angle for s-light would exist. If it so happens that $\varepsilon_a = \varepsilon_b$ while $\mu_a \neq \mu_b$, we will have $\sin\theta = \sqrt{\mu_b/(\mu_a+\mu_b)}$, which leads to $\cos\theta = \sqrt{\mu_a/(\mu_a+\mu_b)}$ and $\tan\theta = \sqrt{\mu_b/\mu_a}$. Once again, this may be written as $\tan\theta = \sqrt{\mu_b\varepsilon_b/\mu_a\varepsilon_a} = n_b/n_a$, as was the case for p-polarized light.
