PhD Qualifying Exam, Fall 2022

Opti 501

System of units: SI (or MKSA)

1) An electromagnetic plane-wave of frequency ω propagates along the unit-vector $\hat{\boldsymbol{\kappa}}$ in free space. The plane-wave's E-field is specified as $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$. Assume that the k-vector is real-valued (i.e., $\hat{\boldsymbol{\kappa}}$ is real), but that the field amplitude $\boldsymbol{E}_0$ is complex; that is, $\boldsymbol{E}_0 = \boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0''$, where $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are a pair of real-valued (but otherwise arbitrary) vectors.

1 pt a) Write the expression that relates $\boldsymbol{k}$ to ω, c (the speed of light in vacuum), and $\hat{\boldsymbol{\kappa}}$.

1 pt b) Invoke Maxwell's first equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$, to show that $\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0 = 0$. Explain why this relation implies that both $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are perpendicular to the unit-vector $\hat{\boldsymbol{\kappa}}$.

2 pts c) Explain how an arbitrary pair of $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ could give rise to an elliptical state of polarization of the plane-wave. Under what circumstances will the plane-wave be linearly polarized? Under what circumstances will the plane-wave be right (or left) circularly polarized?

2 pts d) Use Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, to find an expression for the plane-wave's H-field in terms of ω, $\hat{\boldsymbol{\kappa}}$, $\boldsymbol{E}_0$, and the impedance Z_0 of free space.

2 pts e) Derive expressions for the time-averaged E-field energy-density $\langle\mathcal{E}_E(\boldsymbol{r},t)\rangle$, time-averaged H-field energy-density $\langle\mathcal{E}_H(\boldsymbol{r},t)\rangle$, and time-averaged Poynting vector $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle$ in terms of the parameters $\varepsilon_0, \mu_0, \hat{\boldsymbol{\kappa}}, \boldsymbol{E}_0', \boldsymbol{E}_0''$. (Recall that $\mathcal{E}_E$, $\mathcal{E}_H$, and $\boldsymbol{S}$ must be real-valued.)

2 pts f) Consider the linear superposition of two plane-waves of the same frequency ω, propagating in the same direction $\hat{\boldsymbol{\kappa}}$, but having different polarization states $\boldsymbol{E}_{01}$ and $\boldsymbol{E}_{02}$. Using an arbitrary pair of complex coefficients (α,β), the superposed E-field may be written as

$$\boldsymbol{E}(\boldsymbol{r},t) = (\alpha\boldsymbol{E}_{01} + \beta\boldsymbol{E}_{02})e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}.$$

The two polarization states are said to be mutually orthogonal if, for all values of α and β, the energy-density of the superposition equals the corresponding linear combination of the individual energy densities of the two plane-waves. (This property also extends to the Poynting vector.) Under what circumstances can $\boldsymbol{E}_{01}$ and $\boldsymbol{E}_{02}$ be considered to be mutually orthogonal?

Hint: $\sin(2x) = 2\sin x\cos x$; $\cos(2x) = \cos^2 x - \sin^2 x$; $\sin^2 x = ½(1-\cos 2x)$;

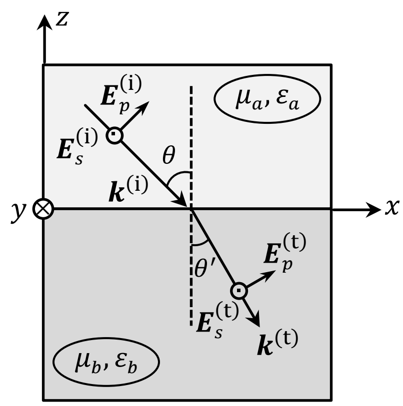
$\cos^2 x = ½(1+\cos 2x)$; $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$; $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$.

PhD Qualifying Exam, Fall 2022

Opti 501

System of units: SI (or MKSA)

2) Consider the flat interface between two linear, isotropic, homogeneous media specified by their relative permeability and permittivity at the incidence frequency, namely, (μ_a, ε_a) for the medium above, and (μ_b, ε_b) for the medium below the interface. These material parameters (i.e., $\mu_a, \mu_b, \varepsilon_a, \varepsilon_b$) are assumed to be real-valued and positive. A homogeneous plane-wave of frequency ω arrives at the interfacial xy-plane; the plane of incidence is xz, the incidence angle is θ, and the E-field components of the incident beam are $\boldsymbol{E}_p^{(\mathrm{i})}$ and $\boldsymbol{E}_s^{(\mathrm{i})}$, as indicated in the figure. The E-field components of the transmitted beam, also a homogeneous plane-wave, are $\boldsymbol{E}_p^{(\mathrm{t})}$ and $\boldsymbol{E}_s^{(\mathrm{t})}$, and the angle between the transmitted k-vector and the surface-normal is θ', as shown.



2 pts a) Invoking the dispersion relation $\boldsymbol{k} \cdot \boldsymbol{k} = (\omega/c)^2 \mu(\omega)\varepsilon(\omega)$, write expressions for the k-vectors of the incident and transmitted plane-waves shown in the above figure. ($c = 1/\sqrt{\mu_0 \varepsilon_0}$ is the speed of light in vacuum.)

2 pts b) Invoking Maxwell's boundary conditions, explain why the transmitted wave has the same frequency ω as the incident wave. What do these boundary conditions reveal about the relation between θ and θ'?

2 pts c) Use Maxwell's third equation, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, to determine both the incident and the transmitted H-field components. (As usual, $\boldsymbol{B} = \mu_0 \mu(\omega) \boldsymbol{H}$; you may use the impedance of free space, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$, to simplify the equations.)

2 pts d) Find the conditions under which the reflected beam for the p-polarized incident light vanishes.

2 pts e) Find the conditions under which the reflected beam for the s-polarized incident light vanishes.