Opti 501 **Final Exam** (12/12/2023) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) In the single-oscillator Lorentz model, absent the Clausius-Mossotti correction, the local material polarization is found to be $\boldsymbol{P}(\boldsymbol{r},t) = \varepsilon_0\chi_e(\omega)\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}$, where the material susceptibility at the excitation frequency ω is given by $\chi_e(\omega) = \omega_p^2/(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega)$. Here, $\omega_p = \sqrt{Nq^2/(\varepsilon_0 m)}$ is the plasma frequency, $\omega_0 = \sqrt{\alpha/m}$ is the resonance frequency, and $\gamma = \beta/m$ is the damping coefficient.

5 pts a) Explain the meaning of the parameters N, q, ε_0, m, α, and β. What are the units of these parameters in the SI (or MKSA) system of units? Also, what are the units of ω_p, ω_0, and γ?

4 pts b) In the context of the Lorentz oscillator model, how does a conduction electron differ from a bound electron? Why is it not necessary to apply the Clausius-Mossotti correction in the case of the conduction electrons?

4 pts c) Recall that $\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t) = \partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t$. The Drude model — same as the Lorentz oscillator model, but for conduction electrons only — defines the electrical conductivity $\sigma(\omega)$ of the material as the proportionality coefficient between the electric current-density and the local electric field; that is, $\boldsymbol{J}(\boldsymbol{r},t) = \sigma(\omega)\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}$. How does Drude's $\sigma(\omega)$ relate to Lorentz's electric susceptibility $\chi_e(\omega)$?

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2 pts **Problem 2**) a) Let a be a real number and $c = c' + \mathrm{i}c''$ a complex number. Clearly, one can write $a\,\mathrm{Re}(c) = \mathrm{Re}(ac) = ac'$, where $\mathrm{Re}(x)$ stands for the real part of x. By the same token, if $c_1 = c_1' + \mathrm{i}c_1''$ and $c_2 = c_2' + \mathrm{i}c_2''$ are two arbitrary complex numbers, one should be able to write $c_1'c_2' = \mathrm{Re}(c_1)\mathrm{Re}(c_2) = \mathrm{Re}[c_1\mathrm{Re}(c_2)]$. Use these elementary properties of real and complex numbers to prove that $\mathrm{Re}(c_1)\mathrm{Re}(c_2) = ½\mathrm{Re}[c_1(c_2 + c_2^*)]$.

3 pts b) Using complex notation, let the electric and magnetic fields at a specific point in spacetime be $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$, respectively. The actual physical fields, of course, are $\mathrm{Re}(\boldsymbol{E})$ and $\mathrm{Re}(\boldsymbol{H})$. The actual (physical) Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$ at the same location, being the result of a nonlinear operation (i.e., multiplication) must be expressed as $\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t)] \times \mathrm{Re}[\boldsymbol{H}(\boldsymbol{r},t)]$. Writing $\boldsymbol{E} = \boldsymbol{E}' + \mathrm{i}\boldsymbol{E}''$ and $\boldsymbol{H} = \boldsymbol{H}' + \mathrm{i}\boldsymbol{H}''$, show that, in general, $\boldsymbol{S} = \mathrm{Re}(\boldsymbol{E}) \times \mathrm{Re}(\boldsymbol{H})$ is *not* the same entity as $\tilde{\boldsymbol{S}} = \mathrm{Re}(\boldsymbol{E} \times \boldsymbol{H})$.

3 pts c) Invoke the results of parts (a) and (b) to show that a correct expression for the actual (i.e., real, physical) Poynting vector is

$$\boldsymbol{S}(\boldsymbol{r},t) = ½\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)] + ½\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}^*(\boldsymbol{r},t)].$$

3 pts d) When the electromagnetic field is monochromatic (i.e., has a single frequency, say, ω_0), one can write the complex fields as $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega_0 t}$ and $\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}(\boldsymbol{r})e^{-\mathrm{i}\omega_0 t}$, with $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ being complex. Show that the Poynting vector of part (c) may now be written as

$$\boldsymbol{S}(\boldsymbol{r},t) = ½\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r})]\cos(2\omega_0 t) + ½\mathrm{Im}[\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r})]\sin(2\omega_0 t) + ½\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})].$$

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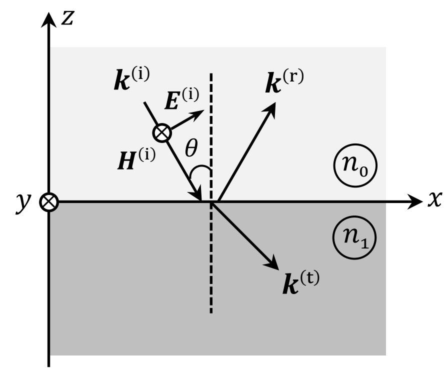
2 pts e) By definition, the time-averaged value of a function $f(t)$ is $\langle f(t)\rangle = (t_2 - t_1)^{-1}\int_{t_1}^{t_2} f(t)\mathrm{d}t$. The oscillatory nature of sinusoidal functions is such that their time-averaged values over a single period $T = 2\pi/(2\omega_0)$ (or over integer-multiples of T) should vanish. Show that, upon period-averaging, $\langle\cos(2\omega_0 t)\rangle = \langle\sin(2\omega_0 t)\rangle = 0$. Conclude that, for monochromatic fields,

$$\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})].$$

Problem 3) A homogeneous, p-polarized plane-wave of frequency ω is obliquely incident at the flat interface between two linear, isotropic, homogeneous media. Both media are non-magnetic; that is, they have $\mu(\omega) = 1$. While the relative permittivity $\varepsilon(\omega)$ of the incidence medium is n_0^2, a real-valued and positive number, that of the transmittance medium is n_1^2, which is generally a complex number, albeit one whose imaginary part is non-negative. The angle of incidence is θ, the speed of light in vacuum is $c = (\mu_0\varepsilon_0)^{-½}$, and the impedance of free space is $Z_0 = (\mu_0/\varepsilon_0)^{½}$.



6 pts a) Invoking the dispersion relation $\boldsymbol{k}\cdot\boldsymbol{k} = (\omega/c)^2\mu(\omega)\varepsilon(\omega)$ as well as Maxwell's 1st and 3rd equations, namely, $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$ and $\boldsymbol{k}\times\boldsymbol{E}_0 = \mu_0\omega\boldsymbol{H}_0$, write the complete expressions for the incident, reflected, and transmitted plane-waves. Incorporate the Fresnel reflection and transmission coefficients for p-polarized light, namely, $\rho_p = E_{0x}^{(\mathrm{r})}/E_{0x}^{(\mathrm{i})}$ and $\tau_p = E_{0x}^{(\mathrm{t})}/E_{0x}^{(\mathrm{i})}$, in the formulas, so that the only field amplitude remaining in your expressions is $E_{0x}^{(\mathrm{i})}$.

4 pts b) Write the two boundary conditions at the interfacial xy-plane in the form of continuity equations for the components of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields that are parallel to the xy-plane.

4 pts c) Recalling that the relative permittivity of a linear, isotropic, homogeneous medium is related to its dielectric susceptibility via $\varepsilon(\omega) = 1 + \chi_e(\omega)$, write an expression for the material polarization $\boldsymbol{P}(\boldsymbol{r},t) = \varepsilon_0\chi_e(\omega)\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ of the transmittance medium. Show that the bound electric charge-density $\rho_{\mathrm{bound}}^{(\mathrm{e})}$ is zero everywhere inside this medium. Proceed to find an expression for the bound electric current-density $\boldsymbol{J}_{\mathrm{bound}}^{(\mathrm{e})}$ inside the transmittance medium.
