Opti 501 **2^{nd} Midterm Exam** (10/31/2023) **Time: 75 minutes**

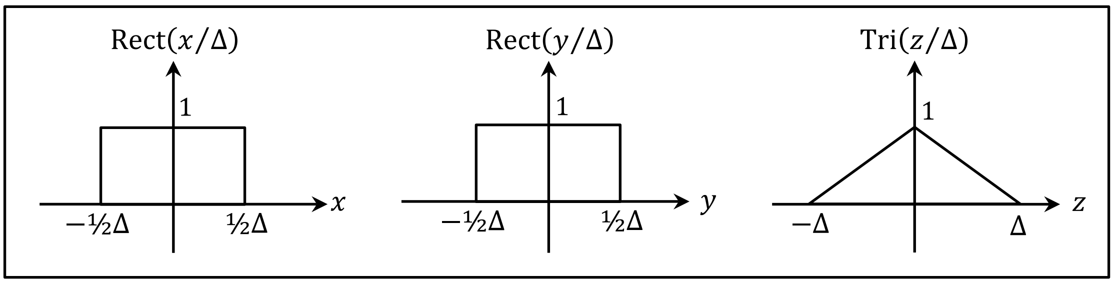
Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) Consider a time-independent polarization distribution centered at the origin of the xyz coordinate system, as follows:

$$\boldsymbol{P}(\boldsymbol{r}) = \frac{p_0\hat{\boldsymbol{z}}}{\Delta^3}\,\mathrm{Rect}\left(\frac{x}{\Delta}\right)\mathrm{Rect}\left(\frac{y}{\Delta}\right)\mathrm{Tri}\left(\frac{z}{\Delta}\right).$$

The dipole moment $p_0\hat{\boldsymbol{z}}$ has units of [coulomb · meter], and the width parameter Δ is in meters.



2 pts a) Find the bound electric charge-density $\rho_{\text{bound}}^{(e)}(\boldsymbol{r})$ corresponding to the polarization $\boldsymbol{P}(\boldsymbol{r})$.

1 pt b) What is the total electric charge in the region above the xy-plane (i.e., in the region $z > 0$)?

1 pt c) What is the total electric charge in the region below the xy-plane (i.e., in the region $z < 0$)?

1 pt d) What is the separation distance between the center of charge above and the center of charge below the xy-plane?

1 pt e) Considering the results obtained in (b), (c), and (d), find the electric dipole moment of the charge distribution in the limit when Δ is sufficiently small.

1 pt f) In the limit when $\Delta \to 0$, the polarization $\boldsymbol{P}(\boldsymbol{r})$ approaches the density of an electric point-dipole located at the origin of the coordinates. Use the Dirac delta-function notation to express the polarization distribution in the limit of $\Delta \to 0$.

Hint: $\boldsymbol{\nabla}\cdot\boldsymbol{V}(\boldsymbol{r}) = \frac{\partial}{\partial x}V_x(r) + \frac{\partial}{\partial y}V_y(r) + \frac{\partial}{\partial z}V_z(r)$.

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2 pts **Problem 2**) a) Write the complete set of Maxwell's partial differential equations. Substitute for $\boldsymbol{D}$ and $\boldsymbol{B}$ in terms of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields, so that the equations retain only the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields — in addition, of course, to the relevant sources $\rho_{\text{free}}(\boldsymbol{r},t)$, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$, $\boldsymbol{P}(\boldsymbol{r},t)$, and $\boldsymbol{M}(\boldsymbol{r},t)$.

1 pt b) Transform the equations obtained in (a) to the Fourier domain. The resulting equations must contain only the Fourier-transformed fields $\boldsymbol{E}(\boldsymbol{k},\omega)$ and $\boldsymbol{H}(\boldsymbol{k},\omega)$ as well as the relevant (Fourier transformed) source fields.

3 pts c) Solve the equations obtained in (b) to find a complete expression for $\boldsymbol{E}(\boldsymbol{k},\omega)$ in terms of the Fourier transformed source fields.

1 pt d) Use the solution obtained for $\boldsymbol{E}(\boldsymbol{k},\omega)$ in conjunction with Maxwell's 3rd equation to arrive at an expression for $\boldsymbol{H}(\boldsymbol{k},\omega)$.

1 pt e) Show that $\boldsymbol{E}(\boldsymbol{k},\omega)$ obtained in (c) satisfies Maxwell's 1st equation.

1 pt f) Show that $\boldsymbol{H}(\boldsymbol{k},\omega)$ obtained in (d) satisfies Maxwell's 4th equation.

Hint: The vector identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ will be helpful.

Problem 3) A uniformly-magnetized solid sphere of radius R sits at the origin of coordinates with magnetization $M_0\hat{\boldsymbol{z}}$ along the z-axis; that is, $\boldsymbol{M}(\boldsymbol{r}) = M_0\hat{\boldsymbol{z}}\,\text{sphere}(r/R)$.

3 pts a) Find the 3-dimensional (3D) Fourier transform of $\boldsymbol{M}(\boldsymbol{r})$, namely, $\boldsymbol{M}(\boldsymbol{k}) = \iiint_{-\infty}^{\infty}\boldsymbol{M}(\boldsymbol{r})e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$.

2 pt b) Considering that the bound electric current-density is given by $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r}) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r})$, find the current-density's 3D Fourier transform $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{k})$.

2 pts c) The bound electric charge-density $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t)$ associated with any magnetization distribution $\boldsymbol{M}(\boldsymbol{r},t)$ is known to be zero. Thus, the charge-current continuity equation for our magnetized sphere becomes $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r}) = 0$. Write the Fourier domain version of this continuity equation, then confirm that $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{k})$ obtained in (b) does in fact satisfy the equation.

2 pts d) Write down the Fourier domain expressions for the scalar and vector potentials in 3D k-sapce, namely, $\psi(\boldsymbol{k})$ and $\boldsymbol{A}(\boldsymbol{k})$, in the Lorenz gauge. (**Note**: You are *not* being asked here to *derive* these expressions, nor are you being asked to carry out an inverse Fourier transformation to find the potentials in the spacetime domain.)

Hint: $\int_{r=0}^{R} r\sin(kr)\,\mathrm{d}r = [\sin(kR) - kR\cos(kR)]/k^2$.

Also, the vector identity $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ could be helpful.
