Opti 501 **2^{nd} Midterm Solutions** (10/31/2023) **Time: 75 minutes**

Problem 1) a) $\rho_{\text{bound}}^{(e)}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}) = -\frac{\partial}{\partial z}P_z(\boldsymbol{r}) = -(p_0/\Delta^3)\text{Rect}\left(\frac{x}{\Delta}\right)\text{Rect}\left(\frac{y}{\Delta}\right)\frac{\text{d}}{\text{d}z}\text{Tri}\left(\frac{z}{\Delta}\right).$

The derivative with respect to z of $\text{Tri}(z/\Delta)$ is zero everywhere except in the interval $-\Delta< z < 0$, where it equals $1/\Delta$, and also in the interval $0 < z < \Delta$, where it equals $-1/\Delta$.

b) In the region above the xy-plane (i.e., $z > 0$), the bound electric charge-density within a $\Delta\times\Delta\times\Delta$ cube equals p_0/Δ^4. The total charge within this cube, therefore, equals p_0/Δ.

c) In the region below the xy-plane (i.e., $z < 0$), the bound electric charge-density within a $\Delta\times\Delta\times\Delta$ cube equals $-p_0/\Delta^4$. The total charge within this cube, therefore, equals $-p_0/\Delta$.

d) The $\Delta\times\Delta\times\Delta$ cubes above and below the xy-plane are centered at $z = \pm\tfrac{1}{2}\Delta$, yielding a separation distance of Δ between the centers of the positive and negative charged cubes.

e) For sufficiently small Δ, the charges p_0/Δ above and $-p_0/\Delta$ below the xy-plane are separated by a distance Δ along the z-axis. The corresponding electric dipole moment thus equals $p_0\hat{\boldsymbol{z}}$.

f) In the limit when $\Delta\to 0$, we have $\Delta^{-1}\text{Rect}(x/\Delta)\to\delta(x)$ and $\Delta^{-1}\text{Rect}(y/\Delta)\to\delta(y)$. Also $\Delta^{-1}\text{Tri}(z/\Delta)\to\delta(z)$. Consequently,

$$\boldsymbol{P}(\boldsymbol{r}) = \frac{p_0\hat{\boldsymbol{z}}}{\Delta^3}\text{Rect}\left(\frac{x}{\Delta}\right)\text{Rect}\left(\frac{y}{\Delta}\right)\text{Tri}\left(\frac{z}{\Delta}\right)\to p_0\hat{\boldsymbol{z}}\delta(x)\delta(y)\delta(z).$$

Problem 2) a) Upon substituting $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)$ in Maxwell's partial differential equations, we arrive at

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t) \quad\to\quad \varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t) - \boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t) = \rho_{\text{total}}^{(e)}(\boldsymbol{r},t), \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) + \partial_t\boldsymbol{D}(\boldsymbol{r},t) \to \boldsymbol{\nabla}\times\boldsymbol{H} = (\boldsymbol{J}_{\text{free}} + \partial_t\boldsymbol{P}) + \partial_t\varepsilon_0\boldsymbol{E} = \boldsymbol{J}_{\text{total}}^{(e)} + \varepsilon_0\partial_t\boldsymbol{E}, \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = -\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t \quad\to\quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{M} - \partial_t\mu_0\boldsymbol{H} = -\boldsymbol{J}_{\text{bound}}^{(m)} - \mu_0\partial_t\boldsymbol{H}, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0 \quad\to\quad \boldsymbol{\nabla}\cdot\mu_0\boldsymbol{H} = -\boldsymbol{\nabla}\cdot\boldsymbol{M} \quad\to\quad \mu_0\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t) = \rho_{\text{bound}}^{(m)}(\boldsymbol{r},t). \tag{4}$$

b) When Fourier transformed, the above equations become

$$\varepsilon_0\text{i}\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega) = \rho_{\text{free}}(\boldsymbol{k},\omega) - \text{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega), \tag{5}$$

$$\text{i}\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega) = \boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \text{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega) - \text{i}\omega\varepsilon_0\boldsymbol{E}(\boldsymbol{k},\omega), \tag{6}$$

$$\not{\text{i}}\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega) = \not{\text{i}}\omega\boldsymbol{M}(\boldsymbol{k},\omega) + \not{\text{i}}\omega\mu_0\boldsymbol{H}(\boldsymbol{k},\omega), \tag{7}$$

$$\mu_0\not{\text{i}}\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega) = -\not{\text{i}}\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega). \tag{8}$$

c) Cross-multiplying Eq.(7) on the left into $\boldsymbol{k}$, then substituting from Eqs.(5) and (6), we find

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)] = \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega) + \mu_0\omega\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)$$

$$\to [\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega)]\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}(\boldsymbol{k},\omega) = \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega) - \text{i}\mu_0\omega[\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \text{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega) - \text{i}\omega\varepsilon_0\boldsymbol{E}(\boldsymbol{k},\omega)]$$

$$\to [-\text{i}\varepsilon_0^{-1}\rho_{\text{free}}(\boldsymbol{k},\omega) - \varepsilon_0^{-1}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k} - k^2\boldsymbol{E}(\boldsymbol{k},\omega) = \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega) - \text{i}\mu_0\omega\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)$$
$$-\mu_0\omega^2\boldsymbol{P}(\boldsymbol{k},\omega) - \mu_0\varepsilon_0\omega^2\boldsymbol{E}(\boldsymbol{k},\omega)$$

$$\rightarrow \qquad \boldsymbol{E}(\boldsymbol{k},\omega)=\frac{\mathrm{i}\mu_0\omega\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\varepsilon_0^{-1}[\rho_{\text{free}}(\boldsymbol{k},\omega)]\boldsymbol{k}+\mu_0\omega^2\boldsymbol{P}(\boldsymbol{k},\omega)-\varepsilon_0^{-1}[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}-\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)}{k^2-(\omega/c)^2}. \tag{9}$$

d) The H-field is now found from Eq.(7) in conjunction with Eq.(9), as follows:

$$\boldsymbol{H}(\boldsymbol{k},\omega)=(\mu_0\omega)^{-1}\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)-\mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega). \tag{10}$$

e) Upon Fourier transforming the charge-current continuity equation, namely,

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{total}}^{(e)}(\boldsymbol{r},t)+\partial\rho_{\text{total}}^{(e)}(\boldsymbol{r},t)/\partial t=0, \tag{11}$$

and recalling that $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)=\partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t+\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)$, we arrive at

$$\cancel{\mathrm{i}}\boldsymbol{k}\cdot\boldsymbol{J}_{\text{total}}^{(e)}(\boldsymbol{k},\omega)-\cancel{\mathrm{i}}\omega\rho_{\text{total}}^{(e)}(\boldsymbol{k},\omega)=0$$

$$\rightarrow\quad \boldsymbol{k}\cdot\left[\boldsymbol{J}_{\text{free}}^{(e)}(\boldsymbol{k},\omega)-\mathrm{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega)+\mu_0^{-1}\mathrm{i}\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)\right]=\omega\left[\rho_{\text{free}}^{(e)}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)\right]. \tag{12}$$

When Fourier transformed, Maxwell's first equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=\rho_{\text{free}}(\boldsymbol{r},t)$, becomes $\mathrm{i}\varepsilon_0\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega)=\rho_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)$. Substitution from Eqs.(9) and (12) now yields

$$\mathrm{i}\varepsilon_0\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega)=\frac{-\mu_0\varepsilon_0\omega\boldsymbol{k}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)+[\rho_{\text{free}}(\boldsymbol{k},\omega)]\boldsymbol{k}\cdot\boldsymbol{k}+\mathrm{i}\mu_0\varepsilon_0\omega^2\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)-\mathrm{i}[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}\cdot\boldsymbol{k}-\mathrm{i}\varepsilon_0\omega\boldsymbol{k}\cdot[\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)]}{k^2-(\omega/c)^2}$$

$$=\frac{-\mu_0\varepsilon_0\omega\boldsymbol{k}\cdot[\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega)+\mu_0^{-1}\mathrm{i}\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)]+[\rho_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]k^2}{k^2-(\omega/c)^2}$$

$$=\frac{-(\omega/c)^2[\rho_{\text{free}}^{(e)}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]+[\rho_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]k^2}{k^2-(\omega/c)^2}$$

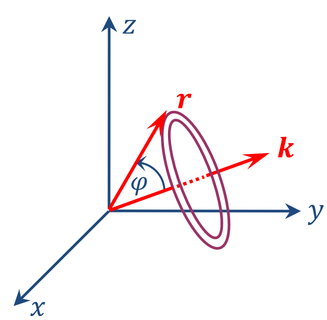
$$=\frac{[k^2-(\omega/c)^2]\,[\rho_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]}{k^2-(\omega/c)^2}=\rho_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega). \tag{13}$$

f) Maxwell's 4th equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0$, leads to $\mu_0\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)=-\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)$ upon Fourier transformation. To confirm the satisfaction of this equation, we dot-multiply Eq.(10) into $\mu_0\boldsymbol{k}$, thus arriving at

$$\mu_0\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)=\omega^{-1}\boldsymbol{k}\cdot[\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)]-\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega) \leftarrow \boxed{\text{use } \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})}$$

$$=\omega^{-1}\overbrace{(\boldsymbol{k}\times\boldsymbol{k})}^{0}\cdot\boldsymbol{E}(\boldsymbol{k},\omega)-\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)=-\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega). \tag{14}$$

Problem 3) The Fourier transform of $\boldsymbol{M}(\boldsymbol{r})$ is computed by 3-dimensional integration in the xyz space, as follows:



$$\mathcal{F}\{\boldsymbol{M}(\boldsymbol{r})\}=\iiint_{-\infty}^{\infty}M_0\hat{\boldsymbol{z}}\,\text{sphere}(r/R)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

$$=M_0\hat{\boldsymbol{z}}\int_{r=0}^{R}\int_{\varphi=0}^{\pi}2\pi r^2\sin\varphi\, e^{-\mathrm{i}kr\cos\varphi}\,\mathrm{d}\varphi\mathrm{d}r$$

$$=2\pi M_0\hat{\boldsymbol{z}}\int_{r=0}^{R}\frac{r^2}{\mathrm{i}kr}e^{-\mathrm{i}kr\cos\varphi}\Big|_{\varphi=0}^{\pi}\mathrm{d}r$$

$$=\left(\frac{2\pi M_0}{\mathrm{i}k}\right)\hat{\boldsymbol{z}}\int_{r=0}^{R}r(e^{\mathrm{i}kr}-e^{-\mathrm{i}kr})\mathrm{d}r$$

$$=\left(\frac{4\pi M_0}{k}\right)\hat{\boldsymbol{z}}\int_{r=0}^{R}r\sin(kr)\,\mathrm{d}r=\frac{4\pi M_0}{k}\hat{\boldsymbol{z}}\left[-\frac{r}{k}\cos(kr)\Big|_{r=0}^{R}+\frac{1}{k}\int_{r=0}^{R}\cos(kr)\,\mathrm{d}r\right]$$

integration by parts

$$= \left(\frac{4\pi M_0}{k^3}\right)[\sin(kR) - kR\cos(kR)]\hat{\boldsymbol{z}}.$$

b) The Fourier transform of the bound electric current-density is readily found to be

$$\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{k}) = \mathcal{F}\{\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{r})\} = \mathcal{F}\{\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r})\} = \mu_0^{-1}\mathrm{i}\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k}) = \frac{\mathrm{i}4\pi M_0}{\mu_0}\left[\frac{\sin(kR) - kR\cos(kR)}{k^3}\right]\boldsymbol{k}\times\hat{\boldsymbol{z}}.$$

c) The continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{r}) = 0$ becomes $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{k}) = 0$ in the Fourier domain. Considering that our $\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{k})$ is orthogonal to the k-vector, one can immediately see that its dot-product into $\boldsymbol{k}$ must vanish. Alternatively, the vector identity $\boldsymbol{k}\cdot(\boldsymbol{k}\times\hat{\boldsymbol{z}}) = \hat{\boldsymbol{z}}\cdot(\boldsymbol{k}\times\boldsymbol{k}) = 0$ can be invoked to arrive at the same conclusion. Needless to say, since the divergence of the curl of any vector field is always zero, the divergence of $\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{r})$ in the xyz-space, being proportional to the curl of $\boldsymbol{M}(\boldsymbol{r})$, should have been expected to vanish all along.

d) In this magnetostatic problem, where $\omega = 0$, we have

$$\psi(\boldsymbol{k}) = \frac{\rho^{(e)}_{\text{bound}}(\boldsymbol{k})}{\varepsilon_0 k^2} = 0,$$

$$\boldsymbol{A}(\boldsymbol{k}) = \frac{\mu_0\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{k})}{k^2} = \mathrm{i}4\pi M_0\left[\frac{\sin(kR) - kR\cos(kR)}{k^4}\right]\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{z}}. \quad \leftarrow \boxed{\hat{\boldsymbol{\kappa}} = \boldsymbol{k}/k}$$

Digression. The vector potential in xyz-space is the inverse Fourier transform of $\boldsymbol{A}(\boldsymbol{k})$; that is,

$$\boldsymbol{A}(\boldsymbol{r}) = \mathcal{F}^{-1}\{\boldsymbol{A}(\boldsymbol{k})\} = -\frac{\mathrm{i}4\pi M_0}{(2\pi)^3}\hat{\boldsymbol{z}}\times\iiint_{-\infty}^{\infty}\hat{\boldsymbol{\kappa}}\left[\frac{\sin(kR) - kR\cos(kR)}{k^4}\right]e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k} \quad \leftarrow \boxed{\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\times\hat{\boldsymbol{\kappa}}}$$

$$= -\frac{\mathrm{i}M_0}{2\pi^2}\hat{\boldsymbol{z}}\times\int_{k=0}^{\infty}\int_{\varphi=0}^{\pi}\hat{\boldsymbol{r}}\cos\varphi\left[\frac{\sin(kR) - kR\cos(kR)}{k^4}\right]e^{\mathrm{i}kr\cos\varphi}2\pi k^2\sin\varphi\,\mathrm{d}\varphi\mathrm{d}k$$

$\uparrow$ $\boxed{\hat{\boldsymbol{r}}\cos\varphi \text{ is the projection of } \hat{\boldsymbol{\kappa}} \text{ onto } \hat{\boldsymbol{r}}}$

$$= -\frac{\mathrm{i}M_0}{\pi}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}\int_{k=0}^{\infty}\left[\frac{\sin(kR) - kR\cos(kR)}{k^2}\right]\int_{\varphi=0}^{\pi}\sin\varphi\cos\varphi\,e^{\mathrm{i}kr\cos\varphi}\mathrm{d}\varphi\mathrm{d}k \quad \leftarrow \boxed{\text{G\&R } \mathbf{3.715}\text{-11}}$$

$$= -\frac{\mathrm{i}M_0}{\pi}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}\int_{k=0}^{\infty}\left[\frac{\sin(kR) - kR\cos(kR)}{k^2}\right]\times\frac{2\mathrm{i}[\sin(kr) - kr\cos(kr)]}{(kr)^2}\mathrm{d}k$$

$$= \frac{2M_0}{\pi r^2}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}\int_{k=0}^{\infty}\frac{[\sin(kR) - kR\cos(kR)][\sin(kr) - kr\cos(kr)]}{k^4}\mathrm{d}k \quad \leftarrow \boxed{\text{see Chapter 3, Problem 22}}$$

$$= \frac{2M_0}{\pi r^2}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}\begin{cases}\pi R^3/6, & r > R;\\ \pi r^3/6, & r \le R.\end{cases} \quad \leftarrow \boxed{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}} = \sin\theta\,\hat{\boldsymbol{\varphi}} \text{ in spherical coordinates}}$$

Consequently,

$$\boldsymbol{A}(r,\theta,\varphi) = \begin{cases}⅓M_0(R^3/r^2)\sin\theta\,\hat{\boldsymbol{\varphi}}, & r > R;\\ ⅓M_0 r\sin\theta\,\hat{\boldsymbol{\varphi}}, & r \le R.\end{cases}$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields may now be obtained from the scalar and vector potentials, as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r}) - \partial_t\boldsymbol{A}(\boldsymbol{r}) = 0.$$

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta\, A_\varphi\right)\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\hat{\boldsymbol{\theta}}$$

$$= \begin{cases} \tfrac{1}{3} M_0 R^3 (2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/r^3, & r > R; \\ \tfrac{2}{3} M_0 (\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \tfrac{2}{3} M_0 \hat{\boldsymbol{z}}, & r \le R. \end{cases}$$

The B-field is seen to be uniform within the spherical magnet, whereas, outside the sphere, it has the same profile as that of a point dipole $\boldsymbol{m} = (4\pi R^3/3) M_0 \hat{\boldsymbol{z}}$ located at the sphere's center.