Opti 501 **1st Midterm Exam** (9/28/2023) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) A plane electromagnetic (EM) wave propagates in free space, where all EM sources are absent; that is, $\rho_{\text{free}}(\boldsymbol{r},t)=0$, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)=0$, $\boldsymbol{P}(\boldsymbol{r},t)=0$, and $\boldsymbol{M}(\boldsymbol{r},t)=0$. The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the plane-wave are specified (in complex notation) as follows:

$$\boldsymbol{E}(\boldsymbol{r},t)=(E_0'\hat{\boldsymbol{x}}+\mathrm{i}E_0''\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]},$$ ← E_0', E_0'' and ω are real

$$\boldsymbol{H}(\boldsymbol{r},t)=(\mathrm{i}H_0''\hat{\boldsymbol{x}}+H_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$$ ← H_0', H_0'' and ω are real

The plane-wave is linearly-polarized if either $E_0'=0$ or $E_0''=0$; it is circularly-polarized if $E_0''=E_0'$ or $E_0''=-E_0'$; otherwise, the plane-wave is said to be elliptically-polarized.

2 pts a) Evaluate the divergence and curl of the plane-wave's $\boldsymbol{E}$ and $\boldsymbol{H}$ fields.

2 pts b) Verify that $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0$

2 pts c) Evaluate $\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t$ and $\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$.

2 pts d) Invoke Maxwell's 2nd equation to relate the components H_0' and H_0'' of the H-field to the components E_0' and E_0'' of the E-field. (Use the identities $c=1/\sqrt{\mu_0\varepsilon_0}$ and $Z_0=\sqrt{\mu_0/\varepsilon_0}$ to simplify your results.)

2 pts e) Repeat part (d) using Maxwell's 3rd equation. Verify that (H_0', H_0'') expressed in terms of (E_0', E_0'') agrees with the result obtained in part (d).

3 pts f) Derive an expression for the plane-wave's Poynting vector. Show that, for a linearly-polarized plane-wave, the Poynting vector has a constant part and a part that varies with the spacetime coordinates (z,t), whereas for a circularly-polarized wave the Poynting vector is constant.

Hint: In Cartesian coordinates,

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}(\boldsymbol{r},t)=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\ \text{ and }\ \boldsymbol{\nabla}\times\boldsymbol{V}(\boldsymbol{r},t)=\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\hat{\boldsymbol{z}}.$$

The trigonometric identities $\sin^2 x=½[1-\cos(2x)]$ and $\cos^2 x=½[1+\cos(2x)]$ can be helpful.

Problem 2) A pair of counter-propagating plane-waves trapped between two parallel, perfectly electrically conducting, flat mirrors is depicted in the figure (below). The plane-waves are monochromatic, have the same frequency ω, and are linearly-polarized along the x-axis. The real-valued $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in the free space region between the mirrors are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=E_0\hat{\boldsymbol{x}}\cos[(\omega/c)z-\omega t]-E_0\hat{\boldsymbol{x}}\cos[(\omega/c)z+\omega t],$$

$$\boldsymbol{H}(\boldsymbol{r},t)=(E_0/Z_0)\hat{\boldsymbol{y}}\cos[(\omega/c)z-\omega t]+(E_0/Z_0)\hat{\boldsymbol{y}}\cos[(\omega/c)z+\omega t].$$

← E_0 and ω are real

The wavelength λ, the angular frequency ω, the oscillation period T, and the frequency f of the plane-waves are related to each other (and to the speed c of light in vacuum) as follows:

$$\lambda=cT=c/f=2\pi c/\omega.$$

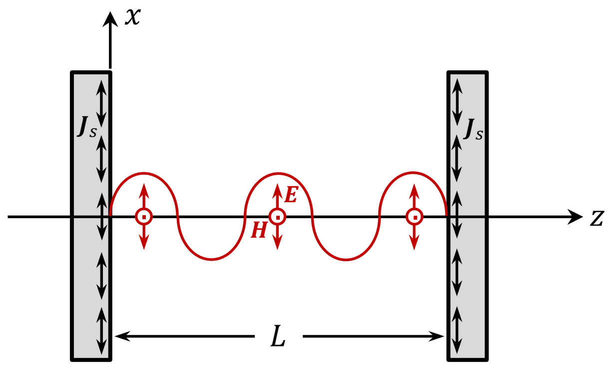
The resonance condition for the cavity is that the distance L separating the (parallel) mirrors must be an integer-multiple of a half wavelength; that is,

$$L = n\lambda/2 = n\pi c/\omega,$$

where n is an arbitrary positive integer. The following trigonometric identities may be used to simplify the expressions of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields within the cavity:

$$\cos a + \cos b = 2\cos[(a+b)/2]\cos[(a-b)/2],$$

$$\cos a - \cos b = -2\sin[(a+b)/2]\sin[(a-b)/2].$$



3 pts a) Verify that the four Maxwell boundary conditions are satisfied at each mirror's inner surface. What are the surface current-densities $\boldsymbol{J}_s(\boldsymbol{r}, t)$ at the inner surfaces of the mirrors?

3 pts b) Use the Lorentz force law to determine the radiation pressure (i.e., force per unit area) exerted by the electromagnetic (EM) field on each mirror.

3 pts c) Write an expression for the energy-density of the EM field in the region between the mirrors. Integrate the energy-density (from $z = 0$ to L) to find the total energy stored within the cavity.

3 pts d) Suppose the mirror on the right-hand side of the cavity moves forward (albeit slowly) by a short distance ΔL along the z-axis. Equate the work done by the radiation force on the mirror with the reduction in the stored energy of the EM field to reveal that $\Delta E_0/E_0 = -\Delta L/L$. (A similar conclusion, of course, can be reached for the magnetic field amplitude.)

Hint: $\int_0^{n\pi} \sin^2\zeta\, \mathrm{d}\zeta = \int_0^{n\pi} \cos^2\zeta\, \mathrm{d}\zeta = n\pi/2.$