Opti 501 **1st Midterm Solutions** (9/28/2023) **Time: 75 minutes**

Problem 1) a) $\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t)=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0+0+0=0.$

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{z}}$$
$$=\{[0-\mathrm{i}E_0''(\mathrm{i}\omega/c)]\hat{\boldsymbol{x}}+[E_0'(\mathrm{i}\omega/c)-0]\hat{\boldsymbol{y}}+[0-0]\hat{\boldsymbol{z}}\}e^{\mathrm{i}[(\omega/c)z-\omega t]}$$
$$=(\omega/c)(E_0''\hat{\boldsymbol{x}}+\mathrm{i}E_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$$

Similarly,

$$\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t)=\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0+0+0=0,$$
$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\hat{\boldsymbol{z}}$$
$$=\{[0-H_0'(\mathrm{i}\omega/c)]\hat{\boldsymbol{x}}+[\mathrm{i}H_0''(\mathrm{i}\omega/c)-0]\hat{\boldsymbol{y}}+[0-0]\hat{\boldsymbol{z}}\}e^{\mathrm{i}[(\omega/c)z-\omega t]}$$
$$=-(\omega/c)(\mathrm{i}H_0'\hat{\boldsymbol{x}}+H_0''\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$$

b) $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=\boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E}+\boldsymbol{P})=\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}=0.$

$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{\nabla}\cdot(\mu_0\boldsymbol{H}+\boldsymbol{M})=\mu_0\boldsymbol{\nabla}\cdot\boldsymbol{H}=0.$

c) $\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t=\varepsilon_0\,\partial\boldsymbol{E}(\boldsymbol{r},t)/\partial t=-\mathrm{i}\omega\varepsilon_0(E_0'\hat{\boldsymbol{x}}+\mathrm{i}E_0''\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$

$\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t=\mu_0\,\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t=-\mathrm{i}\omega\mu_0(\mathrm{i}H_0''\hat{\boldsymbol{x}}+H_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$

d) $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t\;\rightarrow\;-(\omega/c)(\mathrm{i}H_0'\hat{\boldsymbol{x}}+H_0''\hat{\boldsymbol{y}})=-\mathrm{i}\omega\varepsilon_0(E_0'\hat{\boldsymbol{x}}+\mathrm{i}E_0''\hat{\boldsymbol{y}})$

$\rightarrow\;\sqrt{\mu_0\varepsilon_0}(\mathrm{i}H_0'\hat{\boldsymbol{x}}+H_0''\hat{\boldsymbol{y}})=\varepsilon_0(\mathrm{i}E_0'\hat{\boldsymbol{x}}-E_0''\hat{\boldsymbol{y}})\;\rightarrow\;H_0'=E_0'/Z_0$ and $H_0''=-E_0''/Z_0$

$\rightarrow\;\boldsymbol{H}(\boldsymbol{r},t)=(\mathrm{i}H_0''\hat{\boldsymbol{x}}+H_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}=Z_0^{-1}(-\mathrm{i}E_0''\hat{\boldsymbol{x}}+E_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$

e) $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\,\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t\;\rightarrow\;(\omega/c)(E_0''\hat{\boldsymbol{x}}+\mathrm{i}E_0'\hat{\boldsymbol{y}})=\mathrm{i}\omega\mu_0(\mathrm{i}H_0''\hat{\boldsymbol{x}}+H_0'\hat{\boldsymbol{y}})$

$\rightarrow\;\sqrt{\mu_0\varepsilon_0}(E_0''\hat{\boldsymbol{x}}+\mathrm{i}E_0'\hat{\boldsymbol{y}})=\mu_0(-H_0''\hat{\boldsymbol{x}}+\mathrm{i}H_0'\hat{\boldsymbol{y}})\;\rightarrow\;H_0'=E_0'/Z_0$ and $H_0''=-E_0''/Z_0.$

$\rightarrow\;\boldsymbol{H}(\boldsymbol{r},t)=(\mathrm{i}H_0''\hat{\boldsymbol{x}}+H_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}=Z_0^{-1}(-\mathrm{i}E_0''\hat{\boldsymbol{x}}+E_0'\hat{\boldsymbol{y}})e^{\mathrm{i}[(\omega/c)z-\omega t]}.$

This expression for the H-field is seen to be in complete agreement with that obtained in part (d).

f) $\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t)]=\mathrm{Re}\{(E_0'\hat{\boldsymbol{x}}+\mathrm{i}E_0''\hat{\boldsymbol{y}})\{\cos[(\omega/c)z-\omega t]+\mathrm{i}\sin[(\omega/c)z-\omega t]\}\}$

$=E_0'\hat{\boldsymbol{x}}\cos[(\omega/c)z-\omega t]-E_0''\hat{\boldsymbol{y}}\sin[(\omega/c)z-\omega t].$

$\mathrm{Re}[\boldsymbol{H}(\boldsymbol{r},t)]=\mathrm{Re}\{Z_0^{-1}(-\mathrm{i}E_0''\hat{\boldsymbol{x}}+E_0'\hat{\boldsymbol{y}})\{\cos[(\omega/c)z-\omega t]+\mathrm{i}\sin[(\omega/c)z-\omega t]\}\}$

$=Z_0^{-1}E_0''\hat{\boldsymbol{x}}\sin[(\omega/c)z-\omega t]+Z_0^{-1}E_0'\hat{\boldsymbol{y}}\cos[(\omega/c)z-\omega t].$

$$\boldsymbol{S}(\boldsymbol{r},t)=\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t)]\times\mathrm{Re}[\boldsymbol{H}(\boldsymbol{r},t)]=\{E_0'\hat{\boldsymbol{x}}\cos[(\omega/c)z-\omega t]-E_0''\hat{\boldsymbol{y}}\sin[(\omega/c)z-\omega t]\}$$
$$\times Z_0^{-1}\{E_0''\hat{\boldsymbol{x}}\sin[(\omega/c)z-\omega t]+E_0'\hat{\boldsymbol{y}}\cos[(\omega/c)z-\omega t]\}$$

$$= Z_0^{-1}\{E_0'^2 \cos^2[(\omega/c)z - \omega t] + E_0''^2 \sin^2[(\omega/c)z - \omega t]\}\hat{\boldsymbol{z}}$$

$$= \tfrac{1}{2} Z_0^{-1}\{(E_0'^2 + E_0''^2) + (E_0'^2 - E_0''^2)\cos[2(\omega/c)z - 2\omega t]\}\hat{\boldsymbol{z}}.$$

For linearly-polarized light, either $E_0' = 0$ or $E_0'' = 0$. The Poynting vector will then have a constant (time-averaged) value $\langle \boldsymbol{S}(\boldsymbol{r}, t)\rangle = (E_0'^2/2Z_0)\hat{\boldsymbol{z}}$ or $(E_0''^2/2Z_0)\hat{\boldsymbol{z}}$, in addition to a term that varies with (z, t) as $\cos[2(\omega/c)z - 2\omega t]$. The energy flow rate is always in the direction of $\hat{\boldsymbol{z}}$, although it oscillates (as a function of time t at any given z, or as a function of z at any given time) between zero and a maximum value, the maximum being twice the average flow rate.

For circularly-polarized light, $E_0'' = \pm E_0'$ and, therefore, $\boldsymbol{S}(\boldsymbol{r}, t) = (E_0'^2/Z_0)\hat{\boldsymbol{z}}$, which is a constant along the direction of $\hat{\boldsymbol{z}}$, independent of $\boldsymbol{r}$ and t.

Problem 2) a) Using trigonometric identities, we simplify the expressions of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields, as follows:

$$\boldsymbol{E}(\boldsymbol{r}, t) = 2E_0\hat{\boldsymbol{x}}\sin(\omega z/c)\sin(\omega t),$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = 2(E_0/Z_0)\hat{\boldsymbol{y}}\cos(\omega z/c)\cos(\omega t).$$

At $z = 0$, we have $\sin(\omega z/c) = 0$ and, therefore, $\boldsymbol{E}(x, y, z = 0, t) = 0$. Similarly, at $z = L$, we have $\sin(\omega z/c) = \sin(\omega L/c) = \sin(n\pi) = 0$; consequently, $\boldsymbol{E}(x, y, z = L, t) = 0$. Thus, the overall E-field is seen to vanish at both $z = 0$ and $z = L$. This is in agreement with the Maxwell boundary condition that requires the continuity of the tangential E-field at the mirror's surface, where the E-field immediately inside the conductor is zero. As for the magnetic field, we find

$$\boldsymbol{H}(x, y, z = 0, t) = 2(E_0/Z_0)\hat{\boldsymbol{y}}\cos(\omega t).$$

$$\boldsymbol{H}(x, y, z = L, t) = 2(E_0/Z_0)\hat{\boldsymbol{y}}\cos(\omega L/c)\cos(\omega t) = 2(E_0/Z_0)\hat{\boldsymbol{y}}\cos(n\pi)\cos(\omega t)$$
$$= 2(-1)^n(E_0/Z_0)\hat{\boldsymbol{y}}\cos(\omega t).$$

Considering that the H-field inside the conductors is zero, Maxwell's boundary condition requires the existence of a surface current-density $\boldsymbol{J}_s(x, y, z = 0, t) = -2(E_0/Z_0)\hat{\boldsymbol{x}}\cos(\omega t)$ at the front facet of the mirror on the left-hand side. Similarly, the surface current-density at the inner surface of the mirror on the right must be $\boldsymbol{J}_s(x, y, z = L, t) = 2(-1)^n(E_0/Z_0)\hat{\boldsymbol{x}}\cos(\omega t)$.

The perpendicular components of $\boldsymbol{D} = \varepsilon_0\boldsymbol{E}$ and $\boldsymbol{B} = \mu_0\boldsymbol{H}$ are zero everywhere inside the cavity and also within the conducting mirrors. The continuity of $\boldsymbol{D}_\perp$ at the inner surface of each mirror shows that no surface charge-density exists at the surface. Finally, the continuity of $\boldsymbol{B}_\perp$ confirms the satisfaction of the relevant boundary condition at each mirror's inner surface.

b) Recalling that the effective H-field acting on the surface currents is the *average* of the H-fields immediately in front and behind each sheet of surface current, the Lorentz force (per unit area) on the mirrors is given by

$\boldsymbol{B} = \mu_0\boldsymbol{H}$

$$\boldsymbol{F}(x, y, z = 0, t) = \tfrac{1}{2}\boldsymbol{J}_s(x, y, z = 0, t) \times \boldsymbol{B}(x, y, z = 0, t)$$
$$= -(E_0/Z_0)\hat{\boldsymbol{x}}\cos(\omega t) \times 2\mu_0(E_0/Z_0)\hat{\boldsymbol{y}}\cos(\omega t) = -2\varepsilon_0 E_0^2\cos^2(\omega t)\,\hat{\boldsymbol{z}}.$$

$$\boldsymbol{F}(x, y, z = L, t) = \tfrac{1}{2}\boldsymbol{J}_s(x, y, z = L, t) \times \boldsymbol{B}(x, y, z = L, t) = 2\varepsilon_0 E_0^2\cos^2(\omega t)\,\hat{\boldsymbol{z}}.$$

Given that $\langle\cos^2(\omega t)\rangle = \tfrac{1}{2}$, the time-averaged Lorentz force (per unit area) acting on each mirror's surface is found to be $\varepsilon_0 E_0^2$, in a direction that tends to push the mirrors apart.

c) $\mathcal{E}(\boldsymbol{r},t) = ½\varepsilon_0 \boldsymbol{E}\cdot\boldsymbol{E} + ½\mu_0 \boldsymbol{H}\cdot\boldsymbol{H}$

$$= 2\varepsilon_0 E_0^2 \sin^2(\omega z/c)\sin^2(\omega t) + 2\mu_0 (E_0/Z_0)^2 \cos^2(\omega z/c)\cos^2(\omega t)$$

$$= 2\varepsilon_0 E_0^2[\sin^2(\omega z/c)\sin^2(\omega t) + \cos^2(\omega z/c)\cos^2(\omega t)].$$

Integrating the above energy-density from $z = 0$ to $z = L$ now yields the stored EM energy (per unit cross-sectional area) within the cavity. Note that

$$\int_{z=0}^{L} \sin^2(\omega z/c)\,\mathrm{d}z = (c/\omega)\int_{\zeta=0}^{\omega L/c} \sin^2(\zeta)\,\mathrm{d}\zeta = (c/\omega)\int_{\zeta=0}^{n\pi} \sin^2(\zeta)\,\mathrm{d}\zeta = n\pi c/(2\omega) = ½L.$$

Similarly, $\int_{z=0}^{L} \cos^2(\omega z/c)\,\mathrm{d}z = ½L$. We thus find

$$\text{Energy per unit area} = \int_{z=0}^{L} \mathcal{E}(\boldsymbol{r},t)\mathrm{d}z = 2\varepsilon_0 E_0^2[½L\sin^2(\omega t) + ½L\cos^2(\omega t)] = \varepsilon_0 E_0^2 L.$$

d) The mechanical work done by the radiation on the mirror located at $z = L$ when the mirror moves to $z = L + \Delta L$ equals the radiation force times the displacement; that is,

$$\text{Work done per unit area of the mirror} = \varepsilon_0 E_0^2 \Delta L.$$

In the process, the field amplitudes change to $E_0 + \Delta E_0$ and $H_0 + \Delta H_0$ (with $\Delta H_0 = \Delta E_0/Z_0$), so that the overall change in the energy stored within the cavity (per unit cross-sectional area) is

$$\Delta(\text{Energy}) = \Delta(\varepsilon_0 E_0^2 L) = \varepsilon_0(2E_0\Delta E_0)L + \varepsilon_0 E_0^2 \Delta L.$$

The following identity is being used here: $\mathrm{d}[f^2(x)g(x)]/\mathrm{d}x = 2f(x)f'(x)g(x) + f^2(x)g'(x)$.

This change in energy must be equal in magnitude and opposite in sign to the work done on the system by the radiation pressure; that is,

$$\varepsilon_0(2E_0\Delta E_0)L + \varepsilon_0 E_0^2\Delta L = -\varepsilon_0 E_0^2 \Delta L \quad\rightarrow\quad \Delta E_0/E_0 = -\Delta L/L.$$

The fractional decline in the E-field amplitude (and, similarly, in the H-field amplitude) is seen to equal the fractional increase in the length L of the cavity. The same is true of the stored energy; that is,

$$\frac{\Delta(\text{Energy})}{\text{Energy}} = -\frac{\varepsilon_0 E_0^2 \Delta L}{\varepsilon_0 E_0^2 L} = -\frac{\Delta L}{L}.$$

Digression: If we assume that the number n of the nodes inside the cavity (i.e., the number of zero-crossings of the E and H fields) remain unchanged during the expansion, we will have

$$L = n\lambda/2 \quad\rightarrow\quad \Delta\lambda/\lambda = \Delta L/L.$$

Given that $\lambda = 2\pi c/\omega$, we now have $\Delta\lambda = -2\pi c\Delta\omega/\omega^2 = -\lambda(\Delta\omega/\omega)$ and, therefore, $\Delta\lambda/\lambda = -\Delta\omega/\omega$. Thus, the fractional change in the stored energy equals the fractional change in the frequency of the EM radiation inside the cavity. This is consistent with the quantum-optical picture of photons residing in the cavity, each having an energy of $\hbar\omega$. The number of photons during the (slow) expansion of the cavity does not change, but their frequency declines, resulting in a reduced overall energy in proportion to the change in frequency (i.e., Doppler shift).
