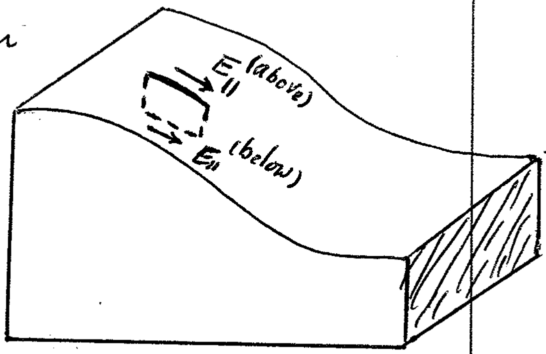
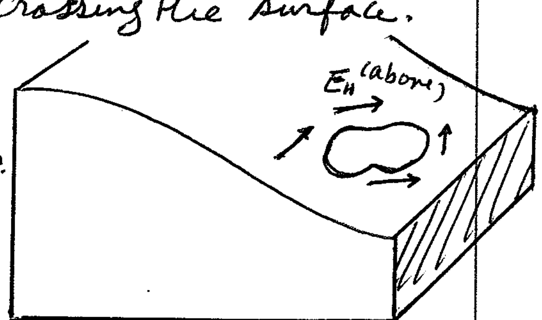
Problem 15)

a) Consider a loop perpendicular to the surface separating two adjacent media. The legs of the loop that penetrate the surface (i.e., vertical legs in the figure) are very short. Therefore, the contributions to $\oint \vec{E}\cdot d\vec{\ell}$ come from the leg above the surface and the leg below the surface.



The flux of the $\vec{B}$-field going through the loop is negligible, because the loop's area is exceedingly small. Thus $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$ implies that $\oint \vec{E}\cdot d\vec{\ell}$ around the loop is zero. This is possible only when $\vec{E}_{\parallel}^{(above)} = \vec{E}_{\parallel}^{(below)}$; in other words, the tangential component of $\vec{E}$ near the surface must remain continuous upon crossing the surface.

Now, consider a loop sitting above the surface and parallel to the surface, as shown. The integral of $\vec{E}_{\parallel}^{(above)}$ taken around this loop, normalized by the loop area, is equal to $-\frac{\partial}{\partial t}\vec{B}_{\perp}^{(above)}$.



If the loop is now moved to a position slightly below the surface, the continuity of $\vec{E}_{\parallel}$ across the surface implies that the loop integral of $\vec{E}_{\parallel}^{(below)}$ will remain the same. Consequently, $-\frac{\partial}{\partial t}\vec{B}_{\perp}^{(below)}$ should be the same as $-\frac{\partial}{\partial t}\vec{B}_{\perp}^{(above)}$. We conclude that, in crossing a surface, the time derivative of $\vec{B}_{\perp}$ should remain intact. As with the continuity of $\vec{E}_{\parallel}$, this is a consequence of Faraday's law, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

Of course, the continuity of $\vec{B}_{\perp}$ is guaranteed by Maxwell's 4th equation, namely, $\vec{\nabla}\cdot\vec{B} = 0$. If one places a thin pill-box at the surface, with the top of the pill-box above the surface, and its bottom half below the surface, then $\vec{\nabla}\cdot\vec{B} = 0$ immediately implies that $\vec{B}_{\perp}^{(above)} = \vec{B}_{\perp}^{(below)}$. The point that we would like to emphasize here is that there is some degree of overlap and redundancy among the various boundary conditions: the continuity of $\vec{E}_{\parallel}$ across the surface is not totally divorced from the continuity of $\vec{B}_{\perp}$, and vice-versa.

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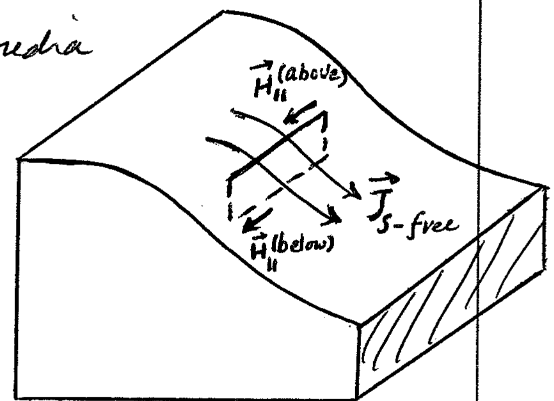
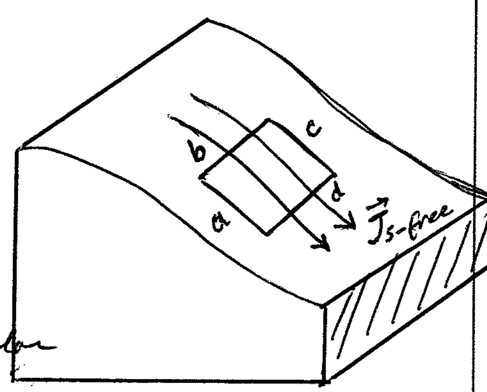
b) The surface separating two adjacent media may be carrying a surface current density, $\vec{J}_{s\text{-}free}$. A small loop placed perpendicular to the surface can be used to evaluate $\oint \vec{H}\cdot d\vec{l}$. The vertical legs of the loop, which cross the surface, are very short.



Using $\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$, we see that the contribution of $\partial\vec{D}/\partial t$ to the loop integral is negligible, because the area of the loop is exceedingly small. However, the contribution of $\vec{J}_{s\text{-}free}$ cannot be ignored, because the current still goes through the loop, no matter how short its vertical legs may be. We conclude that $\vec{H}_{\parallel}^{(above)} - \vec{H}_{\parallel}^{(below)} = \vec{J}_{s\text{-}free}$, when the direction chosen for $\vec{H}_{\parallel}$ is perpendicular to that of $\vec{J}_{s\text{-}free}$. However, for the component of $\vec{H}_{\parallel}$ that is also parallel to the direction of surface current, $\vec{H}_{\parallel}^{(above)} = \vec{H}_{\parallel}^{(below)}$. These statements together form the boundary condition for $\vec{H}_{\parallel}$.

Now, consider a rectangular loop sitting just above the surface and parallel to it. The legs a and c of the loop are parallel to $\vec{J}_{s\text{-}free}$, while the legs b and d are perpendicular to this surface current. The integral of $\vec{H}$ around the loop has contributions from all four legs, a, b, c, and d. Let us move this loop to a position slightly



below the surface (still parallel to the surface). The integral of $\vec{H}$ around this loop has, once again, contributions from the four legs. The contributions of a and c to $\oint \vec{H} \cdot d\vec{\ell}$ are going to be the same, whether the loop is above the surface or below it. This is because $\vec{H}_{\parallel}$ in the direction parallel to $\vec{J}_{s\text{-free}}$ is continuous across the surface. However, the contributions of b and d to the loop integral are going to depend on whether the loop is above or below the surface. The difference of the loop integrals, namely, $\oint_{above} \vec{H} \cdot d\vec{\ell} - \oint_{below} \vec{H} \cdot d\vec{\ell}$, is going to be proportional to $J^{(b)}_{s\text{-free}} - J^{(d)}_{s\text{-free}}$, the diffence between the strengths of the surface current at locations b and d. This difference will be zero if the current $\vec{J}_{s\text{-free}}$ is a divergence-free current at the location of the loop. We conclude that the loop integrals above and below the surface will be the same if $\vec{\nabla} \cdot \vec{J}_{s\text{-free}} = 0$. From the continuity equation, this condition is equivalent to $\partial \sigma_{free} / \partial t = 0$; i.e., no surface charge is being accumulated (or depleted) at the location of the loop.

We now invoke Maxwell's equation $\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$ to conclude that the equality of $\oint \vec{H} \cdot d\vec{\ell}$ above and below the surface, and the equality of $\vec{J}_{\perp\text{-free}}$ above and below the surface, together imply the continuity of $\partial \vec{D}_{\perp} / \partial t$ above and below the surface.

Of course, Maxwell's first equation, $\vec{\nabla} \cdot \vec{D} = \rho_{free}$, tells us that the discontinuity of $\vec{D}_{\perp}$ is equal to the surface charge density, namely, $\vec{D}_{\perp}^{(above)} - \vec{D}_{\perp}^{(below)} = \sigma_{free}^{(Surface)}$. If $\sigma_{free}^{(Surface)}$ does not change with time, then $\frac{\partial}{\partial t} \vec{D}_{\perp}$ will be the same above and below the surface. The point of this exercise, however, was to show that the continuity of $\vec{D}_{\perp}$ across the surface is not entirely unrelated to the continuity of $\vec{H}_{\parallel}$ across the same surface.