Problem 10)

a) $\vec{M}(\vec{r},t) = \dfrac{M_0 R_1}{r}\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\mathrm{Rect}(z/h)\,\hat{r}.$

b) $\vec{J}_b^{(e)}(\vec{r},t) = \dfrac{1}{\mu_0}\vec{\nabla}\times\vec{M}(\vec{r},t) = \dfrac{1}{\mu_0}\dfrac{\partial M_r}{\partial z}\hat{\phi} = \dfrac{M_0 R_1}{\mu_0 r}\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\left[\delta(z+\tfrac{h}{2}) - \delta(z-\tfrac{h}{2})\right]\hat{\phi}$

$\rho_b^{(m)}(\vec{r},t) = -\vec{\nabla}\cdot\vec{M}(\vec{r},t) = \dfrac{-1}{r}\dfrac{\partial}{\partial r}(rM_r) = \dfrac{M_0 R_1}{r}\left[\delta(r-R_2) - \delta(r-R_1)\right]\mathrm{Rect}(z/h) \Longrightarrow$

$\rho_b^{(m)}(\vec{r},t) = M_0\left[\dfrac{R_1}{R_2}\delta(r-R_2) - \delta(r-R_1)\right]\mathrm{Rect}(z/h).$

c) On the inner and outer cylindrical surfaces the velocity is $R_1\omega_0$ and $R_2\omega_0$, respectively. Since $\vec{J}_b^{(m)}(\vec{r},t) = \rho_b^{(m)}(\vec{r},t)\,\vec{V}(\vec{r},t)$, if we write $\vec{V}(\vec{r},t) = r\omega_0\hat{\phi}$, the delta functions present in the expression of $\rho_b^{(m)}$ will ensure that the inner cylinder charges are multiplied with $R_1\omega_0$, while the outer cylinder charges are multiplied with $R_2\omega_0$. We will have:

$\vec{J}_b^{(m)}(\vec{r},t) = \rho_b^{(m)}(\vec{r},t)\,r\omega_0\hat{\phi} = M_0\omega_0\left[R_1\delta(r-R_2) - R_1\delta(r-R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi} \Rightarrow$

$\vec{J}_b^{(m)}(\vec{r},t) = M_0 R_1\omega_0\left[\delta(r-R_2) - \delta(r-R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi}.$

d) Maxwell's 3rd equation: $\vec{\nabla}\times\vec{E}(\vec{r},t) = -\dfrac{\partial}{\partial t}\vec{B}(\vec{r},t) \Rightarrow \vec{\nabla}\times\epsilon_0\vec{E}(\vec{r},t) = -\epsilon_0\dfrac{\partial\vec{M}}{\partial t} - \mu_0\epsilon_0\dfrac{\partial\vec{H}}{\partial t}$

$\Rightarrow \vec{\nabla}\times\vec{D}(\vec{r},t) = -\epsilon_0\left(\dfrac{\partial\vec{M}}{\partial t} - \dfrac{1}{\epsilon_0}\vec{\nabla}\times\vec{P}\right) - \mu_0\epsilon_0\dfrac{\partial\vec{H}(\vec{r},t)}{\partial t}$

Comparison with Maxwell's 2nd equation, $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_{free} + \dfrac{\partial\vec{P}}{\partial t} + \dfrac{1}{\mu_0}\vec{\nabla}\times\vec{M}\right) + \mu_0\epsilon_0\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}$, shows that the bound magnetic current may be defined as $\vec{J}_b^{(m)}(\vec{r},t) = \dfrac{\partial\vec{M}}{\partial t} - \dfrac{1}{\epsilon_0}\vec{\nabla}\times\vec{P}$.

In the present problem $\partial\vec{M}/\partial t = 0$, but $\vec{J}_b^{(m)}$ is associated with $\vec{\nabla}\times\vec{P}$, as follows:

$-\dfrac{1}{\epsilon_0}\vec{\nabla}\times\vec{P}(\vec{r},t) = -\dfrac{1}{\epsilon_0}\vec{\nabla}\times\left\{P_0\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\mathrm{Rect}(z/h)\,\hat{z}\right\} = \dfrac{P_0}{\epsilon_0}\dfrac{\partial}{\partial r}\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi}$

$= (P_0/\epsilon_0)\left[-\delta(r-R_2) + \delta(r-R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi}$

Comparison with $\vec{J}_b^{(m)}(\vec{r},t)$ in part (c) reveals that $P_0 = -\epsilon_0 M_0 R_1\omega_0.$