Solutions **Opti 501**

Problem 9)

a) $\rho_{total}(\vec{r},t) = \rho_{bound}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t)$ ✓

$\vec{J}_{total}(\vec{r},t) = \vec{J}_{bound}(\vec{r},t) = \frac{\partial}{\partial t}\vec{P}(\vec{r},t)$ ✓

b) $\vec{J}(\vec{r},t) = \frac{\partial \vec{P}}{\partial t} = -\epsilon_0 \chi_0 \vec{E}(\vec{r})\omega_0 \sin(\omega_0 t - \phi_0)$

✓ When $\phi_0 = 90°$ we'll have $\vec{J}(\vec{r},t) = \epsilon_0 \chi_0 \vec{E}(\vec{r})\omega_0 \cos(\omega_0 t)$, which is in phase with $\vec{E}(\vec{r},t)$.

c) $\vec{P}(\vec{r},t) = \text{Real}\{\epsilon_0\chi_0 e^{i\phi_0}\vec{E}(\vec{r})e^{-i\omega_0 t}\} = \text{Real}\{\epsilon_0\chi_0[\vec{E}'(\vec{r}) + i\vec{E}''(\vec{r})]e^{-i(\omega_0 t - \phi_0)}\}$

$= \epsilon_0\chi_0 \text{Real}\{[\vec{E}'(\vec{r}) + i\vec{E}''(\vec{r})][\cos(\omega_0 t - \phi_0) - i\sin(\omega_0 t - \phi_0)]\} \Rightarrow$

$$\vec{P}(\vec{r},t) = \epsilon_0\chi_0[\vec{E}'(\vec{r})\cos(\omega_0 t - \phi_0) + \vec{E}''(\vec{r})\sin(\omega_0 t - \phi_0)].$$

$$\vec{J}(\vec{r},t) = \frac{\partial \vec{P}}{\partial t} = \epsilon_0\chi_0\omega_0[-\vec{E}'(\vec{r})\sin(\omega_0 t - \phi_0) + \vec{E}''(\vec{r})\cos(\omega_0 t - \phi_0)].$$

$\vec{D}(\vec{r},t) = \epsilon_0\vec{E}(\vec{r},t) + \vec{P}(\vec{r},t) = \epsilon_0 \text{Real}\{\vec{E}(\vec{r})e^{-i\omega_0 t} + \chi_0 e^{i\phi_0}\vec{E}(\vec{r})e^{-i\omega_0 t}\}$

$= \epsilon_0 \text{Real}\{(1 + \chi_0 e^{i\phi_0})\vec{E}(\vec{r})e^{-i\omega_0 t}\}$

Defining the relative permittivity of the medium $\epsilon e^{i\eta} = 1 + \chi_0 e^{i\phi_0}$, we write

$\vec{D}(\vec{r},t) = \epsilon_0 \text{Real}\{\epsilon\vec{E}(\vec{r})e^{-i(\omega_0 t - \eta)}\} = \epsilon_0\epsilon[\vec{E}'(\vec{r})\cos(\omega_0 t - \eta) + \vec{E}''(\vec{r})\sin(\omega_0 t - \eta)].$

d) $\vec{\nabla}\cdot\vec{D}(\vec{r},t) = \rho_{free}(\vec{r},t) = 0$ ← Maxwell's first equation, in the absence of ρ_{free}, ensures that $\vec{\nabla}\cdot\vec{D} = 0$.

$$\vec{\nabla}\cdot\vec{D}(\vec{r},t) = \epsilon_0\epsilon\,\vec{\nabla}\cdot\vec{E}'(\vec{r})\cos(\omega_0 t-\eta) + \epsilon_0\epsilon\,\vec{\nabla}\cdot\vec{E}''(\vec{r})\sin(\omega_0 t-\eta) = 0 \implies$$

$\Bigg\{$ when $\sin(\omega_0 t-\eta)=0$ we have $\cos(\omega_0 t-\eta)=1 \implies \vec{\nabla}\cdot\vec{E}'(\vec{r})=0$

Similarly, when $\cos(\omega_0 t-\eta)=0 \Rightarrow \sin(\omega_0 t-\eta)=1 \implies \nabla\cdot\vec{E}''(\vec{r})=0$

Consequently, $\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0$ and $\vec{\nabla}\cdot\vec{P}(\vec{r},t)=0$. ✓

Note that $\vec{\nabla}\cdot\vec{P}=0$ implies that within a homogeneous, linear, isotropic medium the density of bound charges is zero, namely, $\rho_{bound}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t)=0$.