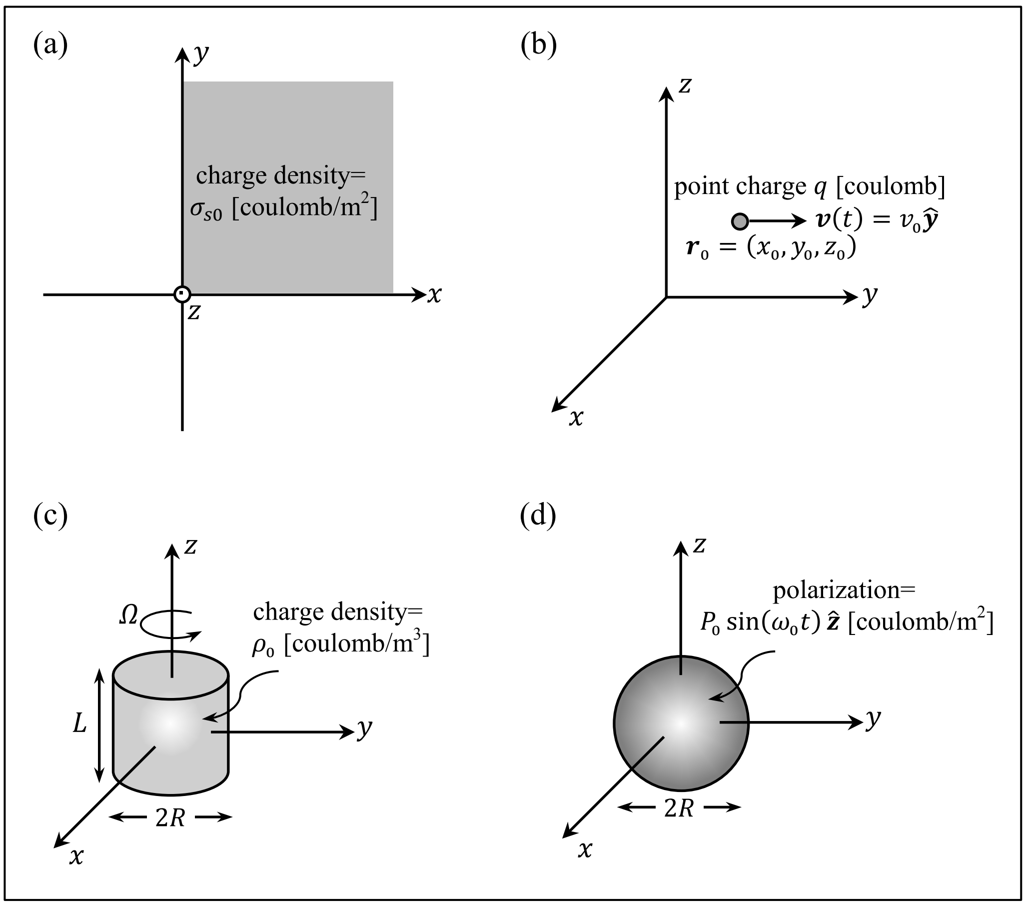
Opti 501 2nd **Midterm Exam** (10/27/2022) **Time: 75 minutes**

**Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.**

Note: Bold symbols represent vectors and vector fields.

Problem 1) Using the standard special functions Step(·), Rect(·), $\delta(\cdot)$, Circ(·), and Sphere(·), write expressions for all the various electromagnetic sources shown in figures (a)-(d) and described in detail below. Answer all accompanying questions.

2 pts a) Thin, semi-infinite sheet of electric charge with constant, uniform surface-charge-density σ_{s0}, sitting in the first quadrant of the xy-plane. The surface-charge-density is zero everywhere except in one quarter of the xy-plane where $x \geq 0$, $y \geq 0$, and $z = 0$; see figure (a).



2 pts b) An electric point-charge q, moving parallel to the y-axis at constant velocity $\boldsymbol{v}(t) = v_0\hat{\boldsymbol{y}}$ and passing through the point $\boldsymbol{r}_0 = x_0\hat{\boldsymbol{x}} + y_0\hat{\boldsymbol{y}} + z_0\hat{\boldsymbol{z}}$ at $t = 0$; see figure (b). Specify both the charge-density $\rho_{\text{free}}(\boldsymbol{r}, t)$ and the current-density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t)$ of this moving charged particle. Verify that the charge-current continuity equation, $\boldsymbol{\nabla} \cdot \boldsymbol{J} + \partial\rho/\partial t = 0$, is satisfied.

2 pts c) Solid cylinder of radius R and height L, centered at the origin of the xyz coordinates and rotating at the constant angular velocity Ω around the z-axis. The uniform electric charge-density of the cylinder is ρ_0; see figure (c). Specify both the charge-density and the current-density of the rotating cylinder. Confirm that the charge-current continuity equation holds.

2 pts d) Uniformly polarized solid sphere of radius R, centered at the origin of the coordinate system, having polarization $\boldsymbol{P}(\boldsymbol{r},t) = P_0 \sin(\omega_0 t)\,\hat{\boldsymbol{z}}$; see figure (d). What is the bound electric charge-density on the sphere's surface? What is the bound electric current-density inside the sphere?

Hint: In the cylindrical coordinate system (ρ, φ, z): $\boldsymbol{\nabla} \cdot \boldsymbol{V}(\boldsymbol{r}) = \dfrac{\partial(\rho V_\rho)}{\rho \partial \rho} + \dfrac{\partial V_\varphi}{\rho \partial \varphi} + \dfrac{\partial V_z}{\partial z}$.

In the spherical coordinate system (r, θ, φ): $\boldsymbol{\nabla} \cdot \boldsymbol{V}(\boldsymbol{r}) = \dfrac{\partial(r^2 V_r)}{r^2 \partial r} + \dfrac{1}{r \sin\theta} \dfrac{\partial(\sin\theta\, V_\theta)}{\partial \theta} + \dfrac{1}{r \sin\theta} \dfrac{\partial(V_\varphi)}{\partial \varphi}$.

Also, the derivative of the unit-sphere function with respect to r is $\mathrm{d}[\text{sphere}(r)]/\mathrm{d}r = -\delta(r-1)$; this is because $\text{sphere}(r)$ drops from the constant value of 1 to the constant value of 0 at $r = 1$.

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2 pts **Problem 2**) a) Write Maxwell's equations in their differential form, with the sources ($\rho_{\text{free}}, \boldsymbol{J}_{\text{free}}$, $\boldsymbol{P}$, $\boldsymbol{M}$) and the fields ($\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$) expressed as functions of the space-time coordinates $(\boldsymbol{r}, t)$.

2 pts b) Write the Maxwell equations described in part (a) in the Fourier domain, with the sources and fields expressed as functions of the Fourier variables $(\boldsymbol{k}, \omega)$.

2 pts c) Eliminate the displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$ from each set of equations that you have written down in (a) and (b). In the end, the only remaining fields must be $\boldsymbol{E}$ and $\boldsymbol{H}$.

(**Note**: You are asked to do this once in the space-time domain, and then again in the Fourier domain.)

2 pts d) Working in the space-time domain only, identify all the sources that appear in the equations obtained in part (c), namely, all free and bound electric and magnetic charge and current densities in your reduced equations. Specify the units (or dimensions) of the various sources.

3 pts e) Solve the Maxwell equations obtained in the Fourier domain in part (c) for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields; that is, find $\boldsymbol{E}(\boldsymbol{k}, \omega)$ and $\boldsymbol{H}(\boldsymbol{k}, \omega)$ as functions of the various sources in the $(\boldsymbol{k}, \omega)$ domain.

Hint: The vector identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ will be helpful.

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6 pts **Problem 3**) Given a uniformly-polarized solid sphere having $\boldsymbol{P}(\boldsymbol{r},t) = P_0\, \text{sphere}(r/R) \cos(\omega_0 t)\,\hat{\boldsymbol{z}}$, find the Fourier transform $\boldsymbol{P}(\boldsymbol{k}, \omega)$ for this radiator of the electromagnetic fields by carrying out the requisite four-dimensional Fourier integral and showing all the details at intermediate steps.

Hint: The following identities will be helpful:

$$\int_{-\infty}^{\infty} e^{\mathrm{i}(\omega \pm \omega_0)t} \mathrm{d}t = 2\pi\delta(\omega \pm \omega_0),$$

$$\int_{\theta=0}^{\pi} \sin\theta\, e^{\mathrm{i}\beta\cos\theta} \mathrm{d}\theta = (2\sin\beta)/\beta,$$

$$\int x \sin(\beta x)\, \mathrm{d}x = [\sin(\beta x) - \beta x \cos(\beta x)]/\beta^2.$$
