Opti 501 **1st Midterm Exam** (9/29/2022) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) The time-independent magnetization distribution in a spherical coordinate system is given by $\boldsymbol{M}(\boldsymbol{r}) = M_0(r/R)^2 e^{-r/R}\hat{\boldsymbol{r}}$, where M_0 and R are positive constants while $\hat{\boldsymbol{r}}$ is the unit-vector in the radial direction.

2 pts a) Find the distribution of the corresponding bound magnetic charge-density, namely, $\rho_{\text{bound}}^{(m)}(\boldsymbol{r})$.

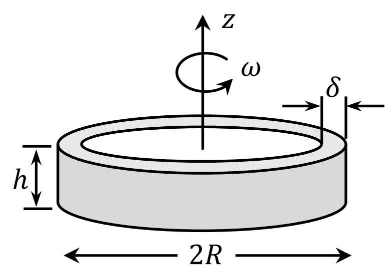
2 pts b) Find the distribution of the corresponding bound electric current-density, namely, $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r})$.

Hint: In spherical coordinates, the divergence and curl of the vector field $\boldsymbol{V}(\boldsymbol{r})$ are given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}(\boldsymbol{r}) = \frac{\partial(r^2V_r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta V_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}.$$

$$\boldsymbol{\nabla}\times\boldsymbol{V}(\boldsymbol{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta V_\varphi)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right]\hat{\boldsymbol{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial(rV_\varphi)}{r\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}.$$

Problem 2) A thin, solid ring of mass m_0, radius R, height h, and width δ is uniformly charged with a constant electric charge-density ρ [coulomb/m^3]. Assume for simplicity that $\delta \ll R$. The ring rotates around the central z-axis at a constant angular velocity ω.



1 pt a) What is the mass-density ζ of the ring?

1 pt b) Find the total electrical charge of the ring.

1 pt c) Find the total mechanical angular momentum $\boldsymbol{\mathcal{L}}$ of the rotating ring.

1 pt d) What is the magnetic dipole-moment $\boldsymbol{m}$ of the rotating ring?

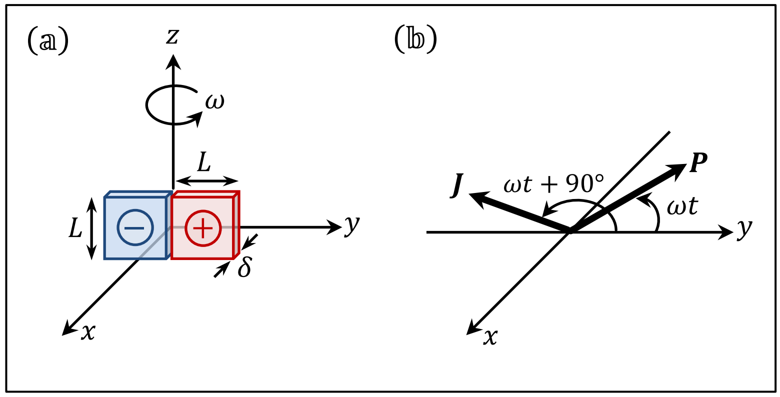
1 pt e) Under what circumstances will $\boldsymbol{\mathcal{L}}$ be anti-parallel to $\boldsymbol{m}$?

4 pts **Problem 3**) a) Write the differential form of Maxwell's first and second equations. Specify the units (or dimensions) for each entity that appears in these equations. Express the units of each entity in terms of the fundamental *SI* units (i.e., meter, kilogram, second, ampere). As a check on your results, confirm that both sides of each equation have the same units.

4 pts b) Repeat part (a) for Maxwell's third and fourth equations, except that this time you must start with the integral form of each equation. Also, identify the relevant mathematical theorem that is invoked in transforming the differential form to the integral form.

Hint: The Lorentz force law $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B})$ may be used to relate the units of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields to the fundamental units.

Problem 4) Consider the side-by-side squares shown in figure (a). The dimensions of both squares are $L \times L \times \delta$, where δ is extremely small. The red square is positively charged, having an electric charge-density $\rho(y)$ that is constant along the x and z axes, but varies in the y direction. The blue square is similarly charged, except that its charge-density is the mirror image of the red square and negative; in other words, the blue square's charge-density is $-\rho(-y)$. Initially, both squares sit in the yz plane, centered at $(x, y, z) = (0, \pm½L, 0)$, as shown.



2 pts a) Write an expression for the electric dipole moment $\boldsymbol{p}$ of the pair in terms of an integral over y.

1 pt b) Using the result of part (a) and assuming that L is sufficiently small, write an expression for the polarization $\boldsymbol{P}$ of the pair.

Suppose now that, starting at $t = 0$, the pair rotates around the z-axis with a constant angular velocity ω. The rotation is counterclockwise as seen from above the xy-plane. The polarization vector $\boldsymbol{P}$, now rotating in the xy-plane, makes an angle ωt with the y-axis, as seen in figure (b).

1 pt c) Using the result of part (b), write an expression for the time-dependent polarization $\boldsymbol{P}(t)$.

1 pt d) Recalling that the local electric current-density $\boldsymbol{J}$ equals the charge-density ρ times the local velocity $\boldsymbol{v}$ (that is, $\boldsymbol{J} = \rho\boldsymbol{v}$), describe the mathematical form of the current-density $\boldsymbol{J}(t)$ at all points on the surface of each square.

1 pt e) Write an expression for the direction of the $\boldsymbol{J}$ vector at time t, and another expression (in the form of an integral) for the total electrical current $I(t)$ that passes through the surfaces of both (red and blue) squares at time t.

1 pt f) Using the results of part (e) and assuming that L is sufficiently small, write an expression for the average current-density $\bar{\boldsymbol{J}}(t)$. (The averaging is done over the pair's total surface area.)

1 pt g) Compare the results of parts (c) and (f) to confirm that $\bar{\boldsymbol{J}}(t) = \mathrm{d}\boldsymbol{P}(t)/\mathrm{d}t$.