Opti 501 **Final Exam** (12/13/2022) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) Recalling that $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}'(\boldsymbol{r},t)+\mathrm{i}\boldsymbol{E}''(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}'(\boldsymbol{r},t)+\mathrm{i}\boldsymbol{H}''(\boldsymbol{r},t)$, compare the following expressions that have been proposed for the Poynting vector:

(i) $\boldsymbol{S}(\boldsymbol{r},t)=\mathrm{Real}\{\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\}$; (ii) $\boldsymbol{S}(\boldsymbol{r},t)=\mathrm{Real}\{\boldsymbol{E}(\boldsymbol{r},t)\}\times\mathrm{Real}\{\boldsymbol{H}(\boldsymbol{r},t)\}$.

3 pts a) Which expression represents the correct form of the Poynting vector? What are the SI units (or dimensions) of $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{S}$ in each of the above expressions? (SI is also called $MKSA$.)

3 pts b) For each expression in (i) and (ii), write $\boldsymbol{S}(\boldsymbol{r},t)$ in terms of the real and imaginary parts of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields. Identify the extraneous term(s) that make one of the expressions incorrect.

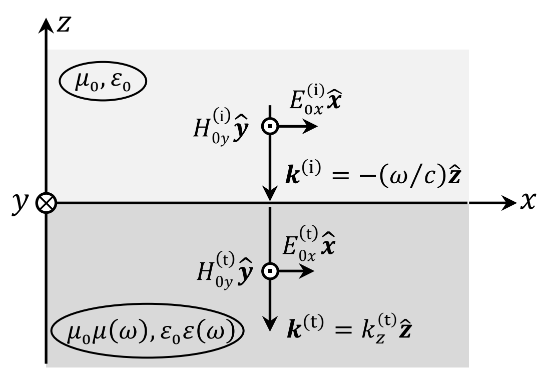
3 pts c) In the case of single-frequency (i.e., monochromatic) electromagnetic fields, one often writes the fields as $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}$ and $\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}(\boldsymbol{r})e^{-\mathrm{i}\omega t}$, where $\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{E}'(\boldsymbol{r})+\mathrm{i}\boldsymbol{E}''(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})=\boldsymbol{H}'(\boldsymbol{r})+\mathrm{i}\boldsymbol{H}''(\boldsymbol{r})$. Write the correct expression of the Poynting vector for this case of a monochromatic field in terms of $\boldsymbol{E}'(\boldsymbol{r})$, $\boldsymbol{E}''(\boldsymbol{r})$, $\boldsymbol{H}'(\boldsymbol{r})$, $\boldsymbol{H}''(\boldsymbol{r})$, $\cos(\omega t)$, and $\sin(\omega t)$.

2 pts d) Use $\boldsymbol{S}(\boldsymbol{r},t)$ derived in part (c) to compute the time-averaged (or period-averaged) Poynting vector, namely, $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=T^{-1}\int_{t_0}^{t_0+T}\boldsymbol{S}(\boldsymbol{r},t)\mathrm{d}t$. Here t_0 is an arbitrary point in time, while $T=2\pi/\omega$ is the period of oscillations.

2 pts e) For the time-averaged $\boldsymbol{S}(\boldsymbol{r},t)$ derived in part (d), show that $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=½\mathrm{Real}\{\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}^*(\boldsymbol{r})\}$.

Hint: $\cos^2(\omega t)=½[1+\cos(2\omega t)]$, $\sin^2(\omega t)=½[1-\cos(2\omega t)]$, $\sin(\omega t)\cos(\omega t)=½\sin(2\omega t)$.

Problem 2) A monochromatic plane-wave of frequency ω arrives at normal incidence at the interface between free space and a metallic medium whose permeability and permittivity are specified as $\mu_0\mu(\omega)$ and $\varepsilon_0\varepsilon(\omega)$. At optical frequencies, one can set $\mu(\omega)=1$ and proceed to take the refractive index of the metallic medium as $n(\omega)=\sqrt{\varepsilon(\omega)}$. Assume that $\varepsilon(\omega)$ can be anywhere in the upper-half of the complex plane and that, therefore, the refractive index can be written as $n(\omega)=n'(\omega)+\mathrm{i}n''(\omega)$ with $n''(\omega)>0$. The incident E-field is aligned with the x-axis, and the Fresnel reflection and transmission coefficients at the metallic surface are $\rho=(1-n)/(1+n)$ and $\tau=2/(1+n)$.



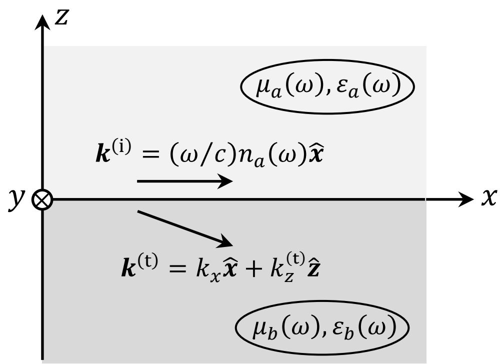
4 pts a) Write expressions for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of incident, reflected, and transmitted plane-waves.

4 pts b) Inside the metallic medium, the bound electric current-density is $\boldsymbol{J}_{\mathrm{bound}}^{(e)}(\boldsymbol{r},t)=\partial\boldsymbol{P}/\partial t=-\mathrm{i}\omega\varepsilon_0\chi_e(\omega)\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t)$, where the material's electric susceptibility is $\chi_e(\omega)=\varepsilon(\omega)-1=n^2(\omega)-1$. Find the bound electrical current throughout the entire depth of the metallic host by integrating $\boldsymbol{J}_{\mathrm{bound}}^{(e)}(\boldsymbol{r},t)$ over the negative-half of the z-axis, i.e., from $z=-\infty$ to 0.

4 pts c) In the limit of a perfect electrical conductor, when $n''\to\infty$, show that the integrated current obtained in part (b) is equal in magnitude (and ⊥ in direction) to the H-field immediately

above the metallic surface. This confirms that the discontinuity of the H-field at the interface between free space and a perfect conductor satisfies Maxwell's second boundary condition.

Problem 3) A homogeneous plane-wave of frequency ω propagates along the x-axis within a semi-infinite, linear, isotropic, homogeneous, transparent medium whose real-valued and positive permeability and permittivity are given by $\mu_0\mu_a(\omega)$ and $\varepsilon_0\varepsilon_a(\omega)$. The (real-valued and positive) refractive index of the medium is, therefore, given by $n_a(\omega) = \sqrt{\mu_a(\omega)\varepsilon_a(\omega)}$.



1 pt a) Use the dispersion relation $\boldsymbol{k}^{(\mathrm{i})} \cdot \boldsymbol{k}^{(\mathrm{i})} = (\omega/c)^2\mu_a\varepsilon_a$ to confirm that $\boldsymbol{k}^{(\mathrm{i})} = (n_a\omega/c)\hat{\boldsymbol{x}}$.

2 pts b) Let $\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = \boldsymbol{E}_0^{(\mathrm{i})}e^{\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r}-\omega t)}$, where $\boldsymbol{E}_0^{(\mathrm{i})} = E_{0x}^{(\mathrm{i})}\hat{\boldsymbol{x}} + E_{0y}^{(\mathrm{i})}\hat{\boldsymbol{y}} + E_{0z}^{(\mathrm{i})}\hat{\boldsymbol{z}}$. Invoke Maxwell's 1st equation, namely, $\boldsymbol{k}\cdot\boldsymbol{D}_0 = 0$, to show that $E_{0x}^{(\mathrm{i})} = 0$.

2 pts c) Let $\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = \boldsymbol{H}_0^{(\mathrm{i})}e^{\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r}-\omega t)}$, where $\boldsymbol{H}_0^{(\mathrm{i})} = H_{0x}^{(\mathrm{i})}\hat{\boldsymbol{x}} + H_{0y}^{(\mathrm{i})}\hat{\boldsymbol{y}} + H_{0z}^{(\mathrm{i})}\hat{\boldsymbol{z}}$. Invoking Maxwell's 3rd equation, namely, $\boldsymbol{k}\times\boldsymbol{E}_0 = \omega\mu_0\mu(\omega)\boldsymbol{H}_0$, find $H_{0x}^{(\mathrm{i})}$, $H_{0y}^{(\mathrm{i})}$, and $H_{0z}^{(\mathrm{i})}$. (**Hint**: $H_{0x}^{(\mathrm{i})}$ turns out to be 0.)

The semi-infinite medium below the xy-plane at $z = 0$ is also linear, isotropic, and homogeneous, with permeability $\mu_0\mu_b(\omega)$ and permittivity $\varepsilon_0\varepsilon_b(\omega)$. However, this medium is not necessarily transparent, which means that μ_b and ε_b could be anywhere on the real axis or in the upper half of the complex plane.

1 pt d) Invoke the generalized version of Snell's law and the knowledge of $k_x^{(\mathrm{i})}, k_y^{(\mathrm{i})}$ to find $k_x^{(\mathrm{t})}, k_y^{(\mathrm{t})}$.

1 pt e) Use the dispersion relation $\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{k}^{(\mathrm{t})} = (\omega/c)^2\mu_b\varepsilon_b$ to find the complete expression of $\boldsymbol{k}^{(\mathrm{t})}$.

2 pts f) Invoke the continuity of $\boldsymbol{E}_\parallel$ at the interfacial xy-plane — a consequence of Maxwell's 3rd equation — to find $E_{0x}^{(\mathrm{t})}, E_{0y}^{(\mathrm{t})}$, the tangential components of the E-field inside the transmittance medium, in terms of the components of the incident E-field.

2 pts g) Invoking the continuity of $\boldsymbol{D}_\perp$ at the interfacial xy-plane — derived from Maxwell's 1st equation — find $E_{0z}^{(\mathrm{t})}$, the z-component of the E-field inside the transmittance medium, in terms of $E_{0z}^{(\mathrm{i})}$ and the material parameters $\varepsilon_a, \varepsilon_b$.

2 pts h) Recalling the continuity of $\boldsymbol{H}_\parallel$ at the interfacial plane — derived from Maxwell's 2nd equation in the absence of surface currents — find $H_{0x}^{(\mathrm{t})}$ and $H_{0y}^{(\mathrm{t})}$, the tangential components of the H-field inside the transmittance medium, in terms of $E_{0z}^{(\mathrm{i})}$ and the material parameters μ_a, ε_a.

2 pts i) Using the continuity of $\boldsymbol{B}_\perp$ at the interfacial plane — a consequence of Maxwell's 4th equation — find $H_{0z}^{(\mathrm{t})}$, the z-component of the H-field inside the transmittance medium, in terms of $E_{0y}^{(\mathrm{i})}$ and the material parameters $\mu_a, \mu_b, \varepsilon_a$.

Assuming that you have done everything correctly up to this point, you will notice that the plane-wave inside the transmittance medium violates at least one of Maxwell's equations; that is, (i) $\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{D}_0^{(\mathrm{t})} \neq 0$, (ii) $\boldsymbol{k}^{(\mathrm{t})}\times\boldsymbol{H}_0^{(\mathrm{t})} \neq -\omega\boldsymbol{D}_0^{(\mathrm{t})}$, (iii) $\boldsymbol{k}^{(\mathrm{t})}\times\boldsymbol{E}_0^{(\mathrm{t})} \neq \omega\boldsymbol{B}_0^{(\mathrm{t})}$, (iv) $\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{B}_0^{(\mathrm{t})} \neq 0$. This is because the scenario depicted in the above figure is physically impossible.
