Problem 1) a) $\rho_{\text{free}}(\boldsymbol{r},t) = \sigma_{s0}\text{Step}(x)\text{Step}(y)\delta(z)$. The units on both sides are [coulomb/m^3].

b) $\rho_{\text{free}}(\boldsymbol{r},t) = q\delta(x-x_0)\delta(y-y_0-v_0t)\delta(z-z_0)$.

$\boldsymbol{J}_{\text{free}}(r,t) = \rho_{\text{free}}(\boldsymbol{r},t)\boldsymbol{v}(t) = qv_0\delta(x-x_0)\delta(y-y_0-v_0t)\delta(z-z_0)\hat{\boldsymbol{y}}$.

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}} + \frac{\partial\rho_{\text{free}}}{\partial t} = \frac{\partial J_{y(\text{free})}}{\partial y} + \frac{\partial\rho_{\text{free}}}{\partial t} = qv_0\delta(x-x_0)\delta'(y-y_0-v_0t)\delta(z-z_0)$$

$\boxed{\partial\delta(y-y_0-v_0t)/\partial t = -v_0\delta'(y-y_0-v_0t)}$ → $-qv_0\delta(x-x_0)\delta'(y-y_0-v_0t)\delta(z-z_0) = 0$.

c) $\rho_{\text{free}}(\boldsymbol{r},t) = \rho_0\text{Circ}(r_\parallel/R)\text{Rect}(z/L)$; $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = \rho_0\underbrace{r_\parallel\Omega}_{\text{linear velocity at distance } r_\parallel \text{ from the } z\text{-axis}}\text{Circ}(r_\parallel/R)\text{Rect}(z/L)\hat{\boldsymbol{\varphi}}$.

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}} + \frac{\partial\rho_{\text{free}}}{\partial t} = \frac{\partial J_{\varphi(\text{free})}}{r_\parallel\partial\varphi} + \frac{\partial\rho_{\text{free}}}{\partial t} = 0 + 0 = 0.$$

d) $\boldsymbol{P}(\boldsymbol{r},t) = P_0\,\text{Sphere}(r/R)\sin(\omega_0 t)\,\hat{\boldsymbol{z}}$.

$$\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\big[P_0\,\text{Sphere}(r/R)\sin(\omega_0 t)\,\overbrace{(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})}^{\hat{\boldsymbol{z}}}\big]$$

$$= -P_0\left\{\frac{\partial[r^2\text{Sphere}(r/R)\cos\theta]}{r^2\partial r} - \frac{1}{r\sin\theta}\frac{\partial[\sin\theta\,\text{Sphere}(r/R)\sin\theta]}{\partial\theta}\right\}\sin(\omega_0 t)$$

$$= -P_0\left\{\frac{[\cancel{2r\,\text{Sphere}(r/R)} - r^2\delta(r-R)]\cos\theta}{r^2} - \cancel{\frac{2\sin\theta\cos\theta\,\text{Sphere}(r/R)}{r\sin\theta}}\right\}\sin(\omega_0 t)$$

$$= P_0\delta(r-R)\cos\theta\sin(\omega_0 t).$$

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t) = \partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t = P_0\omega_0\text{Sphere}(r/R)\cos(\omega_0 t)\,\hat{\boldsymbol{z}}.$$

In the above derivation of the bound charge-density, we have written the radial derivative of $\text{Sphere}(r/R)$ as $-\delta(r-R)$, simply by noting that $\text{Sphere}(r/R)$ drops from the constant value of 1 to the constant value of 0 at $r=R$. Alternatively, one could use the following argument based on the properties of the δ-function and the assertion that $\partial[\text{Sphere}(r)]/\partial r = -\delta(r-1)$:

$$\partial[\text{Sphere}(r/R)]/\partial r = -R^{-1}\delta[(r/R)-1] = -R^{-1}\delta[(r-R)/R] = -\delta(r-R).$$

Digression. Note that the bound charge-density is confined to the sphere's surface at $r=R$. When the surface charge on the upper hemisphere $(0 \le \theta < \pi/2)$ is positive, that on the lower hemisphere $(\pi/2 < \theta \le \pi)$ will be negative, and vice-versa. The maximum and minimum of the charge-density appear at the poles, while at the equator $(\theta = \pi/2)$ the charge-density is always zero. It is also easy now to directly verify the satisfaction of the charge-current continuity equation using the above expressions for $\rho_{\text{bound}}^{(e)}$ and $\boldsymbol{J}_{\text{bound}}^{(e)}$.

Problem 2) a) With $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)$, the standard differential form of Maxwell's equations in the space-time domain are

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t), \quad (1a)$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) + \partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t, \quad (1b)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\,\partial \boldsymbol{B}(\boldsymbol{r},t)/\partial t, \tag{1c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0. \tag{1d}$$

b) The above equations, when transformed into the four-dimensional Fourier domain, become

$$\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{D}(\boldsymbol{k},\omega) = \rho_{\text{free}}(\boldsymbol{k},\omega), \tag{2a}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{H}(\boldsymbol{k},\omega) = \boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \mathrm{i}\omega \boldsymbol{D}(\boldsymbol{k},\omega), \tag{2b}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{k},\omega) = \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{k},\omega), \tag{2c}$$

$$\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{B}(\boldsymbol{k},\omega) = 0. \tag{2d}$$

c_1) Substitution for $\boldsymbol{D}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ in the equations of the space-time domain in (a) yields

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t) - \boldsymbol{\nabla} \cdot \boldsymbol{P}(\boldsymbol{r},t), \tag{3a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r},t) = [\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) + \partial \boldsymbol{P}(\boldsymbol{r},t)/\partial t] + \varepsilon_0 \partial \boldsymbol{E}(\boldsymbol{r},t)/\partial t, \tag{3b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\,\mu_0 \partial \boldsymbol{H}(\boldsymbol{r},t)/\partial t - \partial \boldsymbol{M}(\boldsymbol{r},t)/\partial t, \tag{3c}$$

$$\mu_0 \boldsymbol{\nabla} \cdot \boldsymbol{H}(\boldsymbol{r},t) = -\,\boldsymbol{\nabla} \cdot \boldsymbol{M}(\boldsymbol{r},t). \tag{3d}$$

c_2) Substituting $\varepsilon_0 \boldsymbol{E}(\boldsymbol{k},\omega) + \boldsymbol{P}(\boldsymbol{k},\omega)$ for $\boldsymbol{D}(\boldsymbol{k},\omega)$ and $\mu_0 \boldsymbol{H}(\boldsymbol{k},\omega) + \boldsymbol{M}(\boldsymbol{k},\omega)$ for $\boldsymbol{B}(\boldsymbol{k},\omega)$ in the Fourier domain equations in part (b) yields

$$\mathrm{i}\boldsymbol{k} \cdot \varepsilon_0 \boldsymbol{E}(\boldsymbol{k},\omega) = \rho_{\text{free}}(\boldsymbol{k},\omega) - \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{P}(\boldsymbol{k},\omega), \tag{4a}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{H}(\boldsymbol{k},\omega) = [\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \mathrm{i}\omega \boldsymbol{P}(\boldsymbol{k},\omega)] - \mathrm{i}\omega\varepsilon_0 \boldsymbol{E}(\boldsymbol{k},\omega), \tag{4b}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{k},\omega) = \mathrm{i}\omega\mu_0 \boldsymbol{H}(\boldsymbol{k},\omega) + \mathrm{i}\omega \boldsymbol{M}(\boldsymbol{k},\omega), \tag{4c}$$

$$\mathrm{i}\boldsymbol{k} \cdot \mu_0 \boldsymbol{H}(\boldsymbol{k},\omega) = -\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{M}(\boldsymbol{k},\omega). \tag{4d}$$

d) The free and bound charge and current densities can be read from the expanded version of Maxwell's space-time equations given in (c_1), as follows:

Free electric charge-density: $\rho_{\text{free}}(\boldsymbol{r},t)$ [coulomb/m^3], (5a)

Bound electric charge-density: $\rho^{(e)}_{\text{bound}}(\boldsymbol{r},t) = -\boldsymbol{\nabla} \cdot \boldsymbol{P}(\boldsymbol{r},t)$ [coulomb/m^3], (5b)

Free electric current-density: $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ [ampere/m^2], (5c)

Bound electric current-density: $\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{r},t) = \partial \boldsymbol{P}(\boldsymbol{r},t)/\partial t$ [ampere/m^2], (5d)

Bound magnetic charge-density: $\rho^{(m)}_{\text{bound}}(\boldsymbol{r},t) = -\boldsymbol{\nabla} \cdot \boldsymbol{M}(\boldsymbol{r},t)$ [weber/m^3], (5e)

Bound magnetic current-density: $\boldsymbol{J}^{(m)}_{\text{bound}}(\boldsymbol{r},t) = \partial \boldsymbol{M}(\boldsymbol{r},t)/\partial t$ [weber/(m$^2 \cdot$ sec)]. (5f)

e) Start by cross-multiplying Maxwell's 3rd equation (in the Fourier domain) into $\boldsymbol{k}$, then use the vector identity to reduce the triple-cross-product, then use Maxwell's 1st and 2nd equations (again, in the Fourier domain) to substitute for $\boldsymbol{k} \cdot \boldsymbol{E}(\boldsymbol{k},\omega)$ and $\boldsymbol{k} \times \boldsymbol{H}(\boldsymbol{k},\omega)$. You will find

$$\boldsymbol{k} \times [\boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{k},\omega)] = \omega\mu_0 \boldsymbol{k} \times \boldsymbol{H}(\boldsymbol{k},\omega) + \omega \boldsymbol{k} \times \boldsymbol{M}(\boldsymbol{k},\omega)$$

$$\rightarrow \quad (\boldsymbol{k} \cdot \boldsymbol{E})\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{k})\boldsymbol{E} = \mu_0\omega(-\mathrm{i}\boldsymbol{J}_{\text{free}} - \omega \boldsymbol{P} - \varepsilon_0 \omega \boldsymbol{E}) + \omega \boldsymbol{k} \times \boldsymbol{M}$$

$$\rightarrow \quad \varepsilon_0^{-1}(-\mathrm{i}\rho_{\text{free}} - \boldsymbol{k} \cdot \boldsymbol{P})\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{k})\boldsymbol{E} = -\mu_0\omega(\mathrm{i}\boldsymbol{J}_{\text{free}} + \omega \boldsymbol{P}) - \mu_0\varepsilon_0\omega^2 \boldsymbol{E} + \omega \boldsymbol{k} \times \boldsymbol{M}$$

$$\rightarrow \quad (\boldsymbol{k}\cdot\boldsymbol{k} - \mu_0\varepsilon_0\omega^2)\boldsymbol{E} = -\mathrm{i}\varepsilon_0^{-1}\rho_{\text{free}}\boldsymbol{k} + \mathrm{i}\mu_0\omega\boldsymbol{J}_{\text{free}} - \varepsilon_0^{-1}(\boldsymbol{k}\cdot\boldsymbol{P})\boldsymbol{k} + \mu_0\omega^2\boldsymbol{P} - \omega\boldsymbol{k}\times\boldsymbol{M}$$

$$\rightarrow \quad \boldsymbol{E}(\boldsymbol{k},\omega) = \frac{-\mathrm{i}\rho_{\text{free}}(\boldsymbol{k},\omega)\boldsymbol{k} + \mathrm{i}(\omega/c^2)\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - [\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k} + (\omega/c)^2\boldsymbol{P}(\boldsymbol{k},\omega) - \varepsilon_0\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)}{\varepsilon_0[k^2 - (\omega/c)^2]}. \qquad (6)$$

Substituting the above expression of $\boldsymbol{E}(\boldsymbol{k},\omega)$ into Maxwell's 3rd equation yields the H-field, as follows:

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{k},\omega) &= (\mu_0\omega)^{-1}\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega) - \mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega)\\
&= \frac{\mathrm{i}(\omega/c^2)\boldsymbol{k}\times\boldsymbol{J}_{\text{free}} + (\omega/c)^2\boldsymbol{k}\times\boldsymbol{P} - \varepsilon_0\omega\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{M})}{(\omega/c^2)[k^2-(\omega/c)^2]} - \mu_0^{-1}\boldsymbol{M}\\
&= \frac{\mathrm{i}(\omega/c^2)\boldsymbol{k}\times\boldsymbol{J}_{\text{free}} + (\omega/c)^2\boldsymbol{k}\times\boldsymbol{P} - \varepsilon_0\omega[(\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k} - k^2\boldsymbol{M}] - \varepsilon_0\omega[k^2-(\omega/c)^2]\boldsymbol{M}}{(\omega/c^2)[k^2-(\omega/c)^2]}\\
&= \frac{\mathrm{i}(\omega/c^2)\boldsymbol{k}\times\boldsymbol{J}_{\text{free}} + (\omega/c)^2\boldsymbol{k}\times\boldsymbol{P} - \varepsilon_0\omega(\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k} + \varepsilon_0\omega(\omega/c)^2\boldsymbol{M}}{(\omega/c^2)[k^2-(\omega/c)^2]}\\
&= \frac{\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}} + \omega\boldsymbol{k}\times\boldsymbol{P} - \mu_0^{-1}(\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k} + \mu_0^{-1}(\omega/c)^2\boldsymbol{M}}{k^2-(\omega/c)^2}. \qquad (7)
\end{aligned}$$

Digression. Note that Eq.(6) satisfies Maxwell's 1st equation, namely,

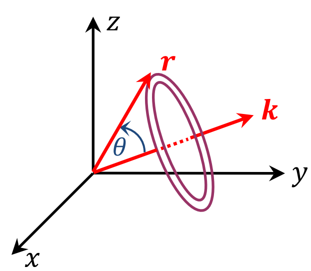
$$\mathrm{i}\boldsymbol{k}\cdot(\varepsilon_0\boldsymbol{E} + \boldsymbol{P}) = \frac{\rho_{\text{free}}\boldsymbol{k}\cdot\boldsymbol{k} - (\omega/c^2)\boldsymbol{k}\cdot\boldsymbol{J}_{\text{free}} - \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{P})\boldsymbol{k}\cdot\boldsymbol{k} + \mathrm{i}(\omega/c)^2\boldsymbol{k}\cdot\boldsymbol{P} - \cancel{\mathrm{i}\varepsilon_0\omega\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{M})}}{k^2-(\omega/c)^2} + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}$$

$$\boxed{\boldsymbol{k}\cdot\boldsymbol{J}_{\text{free}} = \omega\rho_{\text{free}}} \rightarrow = \frac{[k^2-(\omega/c)^2]\rho_{\text{free}} - \mathrm{i}[k^2-(\omega/c)^2]\boldsymbol{k}\cdot\boldsymbol{P}}{k^2-(\omega/c)^2} + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P} = \rho_{\text{free}}(\boldsymbol{k},\omega).$$

Similarly, Eq.(7) satisfies Maxwell's 4th equation; that is,

$$\boldsymbol{k}\cdot(\mu_0\boldsymbol{H} + \boldsymbol{M}) = \frac{\cancel{\mu_0(\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}} + \omega\boldsymbol{k}\times\boldsymbol{P})\cdot\boldsymbol{k}} - (\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k}\cdot\boldsymbol{k} + (\omega/c)^2\boldsymbol{k}\cdot\boldsymbol{M}}{k^2-(\omega/c)^2} + \boldsymbol{k}\cdot\boldsymbol{M} = 0.$$

Problem 3) The integral over t is separable from the integral over $\boldsymbol{r}$. The spherical symmetry of the system allows the integral over $\boldsymbol{r}$ to be reduced to an integral over a thin ring centered on $\boldsymbol{k}$, followed by integration over the polar angle θ (relative to $\boldsymbol{k}$) from 0 to π and, finally, an integral over the radial coordinate r from 0 to ∞.



$$\begin{aligned}
\boldsymbol{P}(\boldsymbol{k},\omega) &= \int_{-\infty}^{\infty} P_0\hat{\boldsymbol{z}}\,\text{sphere}(r/R)\cos(\omega_0 t)\, e^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\mathrm{d}\boldsymbol{r}\mathrm{d}t\\
&= \tfrac{1}{2}P_0\hat{\boldsymbol{z}}\int_{-\infty}^{\infty}\text{sphere}(r/R)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}\int_{-\infty}^{\infty}(e^{\mathrm{i}\omega_0 t} + e^{-\mathrm{i}\omega_0 t})e^{\mathrm{i}\omega t}\mathrm{d}t\\
&= \tfrac{1}{2}P_0\hat{\boldsymbol{z}}\Big[\int_{r=0}^{R}\int_{\theta=0}^{\pi} e^{-\mathrm{i}kr\cos\theta}(2\pi r^2\sin\theta)\mathrm{d}\theta\mathrm{d}r\Big][2\pi\delta(\omega+\omega_0) + 2\pi\delta(\omega-\omega_0)]\\
&= 2\pi^2P_0\hat{\boldsymbol{z}}\Big[\int_{r=0}^{R}(r^2/\mathrm{i}kr)\int_{\theta=0}^{\pi}\mathrm{i}kr\sin\theta\, e^{-\mathrm{i}kr\cos\theta}\mathrm{d}\theta\mathrm{d}r\Big][\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]\\
&= 2\pi^2P_0\hat{\boldsymbol{z}}\Big[\int_{r=0}^{R}(r/\mathrm{i}k)e^{-\mathrm{i}kr\cos\theta}\Big|_{\theta=0}^{\pi}\mathrm{d}r\Big][\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]\\
&= (4\pi^2P_0\hat{\boldsymbol{z}}/k)\Big[\int_{r=0}^{R} r\sin(kr)\,\mathrm{d}r\Big][\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]\\
&= (4\pi^2P_0\hat{\boldsymbol{z}}/k)\Big[-(r/k)\cos(kr)\Big|_{r=0}^{R} + k^{-1}\int_{r=0}^{R}\cos(kr)\,\mathrm{d}r\Big][\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]
\end{aligned}$$

$$= (4\pi^2 P_0 \hat{\boldsymbol{z}}/k^3)[\sin(kR) - kR\cos(kR)][\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$

Digression. For a sufficiently small sphere radius R, we will have $\sin(kR) \cong kR - (kR)^3/3!$ and $\cos(kR) \cong 1 - (kR)^2/2!$. Consequently, $\boldsymbol{P}(\boldsymbol{k}, \omega) \cong (4\pi^2 R^3 P_0 \hat{\boldsymbol{z}}/3)[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$, which can be further simplified by noting that the sphere's overall dipole moment is $\boldsymbol{p}_0 = (4\pi R^3/3)P_0\hat{\boldsymbol{z}}$.
