Opti 501 **1st Midterm Solutions** (9/29/2022) **Time: 75 minutes**

Problem 1)

a) $$\rho_{\text{bound}}^{(m)}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r}) = -\frac{\partial(r^2M_r)}{r^2\partial r} = -(M_0/R^2)\frac{\mathrm{d}(r^4e^{-r/R})}{r^2\mathrm{d}r}$$

$$= -(M_0/R^2)\frac{4r^3e^{-r/R}-(r^4/R)e^{-r/R}}{r^2} = (M_0/R^3)(r^2-4Rr)e^{-r/R} \quad [\text{weber/m}^3].$$

b) $$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r}) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r}) = \mu_0^{-1}\left(\frac{1}{r\sin\theta}\frac{\partial M_r}{\partial\varphi}\hat{\boldsymbol{\theta}} - \frac{\partial M_r}{r\partial\theta}\hat{\boldsymbol{\varphi}}\right) = 0 \qquad [\text{ampere/m}^2].$$

Note that $\boldsymbol{M}(\boldsymbol{r})$ is zero at the origin (i.e., at $r=0$), rises to its maximum value at $r=2R$, and from there declines to zero as $r\to\infty$. The corresponding magnetic charge-density $\rho_{\text{bound}}^{(m)}(\boldsymbol{r})$ is zero at $r=0$, has a minimum at $r=(3-\sqrt{5})R$, returns to zero at $r=4R$, reaches its maximum value at $r=(3+\sqrt{5})R$, then declines to zero as $r\to\infty$. In contrast, the bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r})$ is zero everywhere, as the various microscopic loops of current cancel each other out.

Problem 2) a) mass-density $\zeta = m_0/\text{volume} \cong m_0/(2\pi Rh\delta)$ [kg/m^3].

b) Total charge Q = volume × charge-density $\cong (2\pi Rh\delta)\rho$ [coulomb].

c) Given that the linear mechanical momentum of a small volume element $\mathrm{d}v$ of the ring is $\boldsymbol{p}\mathrm{d}v = (\zeta\mathrm{d}v)R\omega\hat{\boldsymbol{\varphi}}$, we will have $\boldsymbol{\mathcal{L}} = \int_{\text{ring}}(\boldsymbol{r}\times\boldsymbol{p})\mathrm{d}v \cong \int_{\text{ring}}(R\hat{\boldsymbol{r}}_\parallel\times\zeta R\omega\hat{\boldsymbol{\varphi}})\mathrm{d}v = m_0R^2\omega\hat{\boldsymbol{z}}$. The units of $\boldsymbol{\mathcal{L}}$ are [kg · m^2/sec]. (Note that, in general, $\boldsymbol{r} = R\hat{\boldsymbol{r}}_\parallel + z\hat{\boldsymbol{z}}$. However, $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{r}}_\parallel$ integrates to zero, which is why it has been dropped from the preceding equation.)

d) $\boldsymbol{m} \cong \mu_0(\rho R\omega)(h\delta)(\pi R^2)\hat{\boldsymbol{z}} = ½\mu_0QR^2\omega\hat{\boldsymbol{z}}$ [henry · ampere · meter = weber · meter].

current-density | cross-section | loop area

e) The magnetic dipole moment $\boldsymbol{m}$ will be anti-parallel to the ring's angular momentum $\boldsymbol{\mathcal{L}}$ if the rotating charge Q happens to be negative.

Problem 3) a) (i) $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_{\text{free}}(\boldsymbol{r},t)$, where $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$.

The units of polarization $\boldsymbol{P}$ are coulomb/m^2, with those of charge, coulomb, being ampere · sec.

The units of displacement $\boldsymbol{D}$ are the same as those of $\boldsymbol{P}$, namely, ampere · sec/m^2.

The units of ρ_{free} are coulomb/m^3, i.e., ampere · sec/m^3.

The units of $\boldsymbol{E}$ are volt/m. Invoking the Lorentz force law, $\boldsymbol{f} = q\boldsymbol{E}$, reveals the units of $\boldsymbol{E}$ to be the same as those of force (kg · m/sec^2) divided by the units of electrical charge (coulomb). Therefore, "volt" is kg · m^2/(ampere · sec^3).

The units of ε_0 are farad/m. For a capacitor having capacitance C, total (positive) charge Q, and voltage V, we have $Q = CV$, with the units of Q, C, and V being coulomb, farad, and volt, respectively. Consequently, farad = coulomb/volt, which makes the units of $\varepsilon_0\boldsymbol{E}$ equal to coulomb/m^2, consistent with those of $\boldsymbol{P}$ and $\boldsymbol{D}$. Substitution for coulomb and volt now yields the units of ε_0 in terms of the fundamental MKSA units as ampere^2 · sec^4/ (kg · m^3).

Digression. An alternative way of expressing farad in terms of the fundamental units is by way of the formula $\mathcal{E} = ½\varepsilon_0 E^2$ for the energy-density of the E-field, where the units of $\mathcal{E}$ are joule/m^3. Thus, kg · m^2/(sec^2 · m^3) = (farad/m) · (volt/m)^2, which yields farad = kg · m^2/(volt^2 · sec^2) = ampere^2 · sec^4/(kg · m^2).

In Maxwell's first equation, the divergence operator divides the units of $\boldsymbol{D}$ by the units of length, namely, meter. Thus, the left-hand side of the equation has units of coulomb/m^3, in agreement with those of ρ_{free} on the right-hand side.

(ii) $$\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) + \partial \boldsymbol{D}(\boldsymbol{r}, t)/\partial t.$$

The units of $\boldsymbol{J}_{\text{free}}$ are ampere/m^2, which are the same as those of $\partial \boldsymbol{D}/\partial t$, since $\boldsymbol{D}$ has units of coulomb/m^2, which, upon differentiation with respect to time (i.e., division of the units by sec), become coulomb/(m^2 · sec) = ampere/m^2.

The units of $\boldsymbol{H}$ are ampere/m. Considering that the curl operation ($\boldsymbol{\nabla} \times$) involves differentiation with respect to spatial coordinates (x, y, z), which have the units of length (i.e., meter), the left-hand side of Maxwell's 2$^{\text{nd}}$ equation has units of ampere/m^2, in agreement with those of the right-hand side.

b) (iii) $\oint_{\text{closed loop}} \boldsymbol{E}(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\text{surface}} \boldsymbol{B}(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{s}$, where $\boldsymbol{B}(\boldsymbol{r}, t) = \mu_0 \boldsymbol{H}(\boldsymbol{r}, t) + \boldsymbol{M}(\boldsymbol{r}, t)$.

This integral form of Maxwell's 3$^{\text{rd}}$ equation is obtained from the corresponding differential form, namely, $\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\partial \boldsymbol{B}(\boldsymbol{r}, t)/\partial t$, via the application of the Stokes theorem.

On the left-hand side of the equation, the units of $\boldsymbol{E}$ are volt/m, which, upon multiplication by the units of $\mathrm{d}\boldsymbol{\ell}$ (namely, meter) become volt = kg · m^2/(ampere · sec^3).

The units of $\boldsymbol{B}$ and $\boldsymbol{M}$ are weber/m^2. From the Lorentz force law $\boldsymbol{f} = q\boldsymbol{V} \times \boldsymbol{B}$, we find that weber/m^2 = newton/(coulomb · m/sec), which yields weber = kg · m^2/(ampere · sec^2).

The units of μ_0 are henry/m. Considering that the speed of light in vacuum is $c = 1/\sqrt{\mu_0 \varepsilon_0}$, we arrive at (henry/m) · (farad/m) = (sec/m)^2, yielding henry = sec^2/farad = kg · m^2/(ampere^2 · sec^2). Multiplying the units of μ_0 (i.e., henry/m) into those of $\boldsymbol{H}$ (i.e., ampere/m) now yields the units of $\mu_0 \boldsymbol{H}$ as kg/(ampere · sec^2), consistent with the units of $\boldsymbol{B}$ and $\boldsymbol{M}$ (i.e., weber/m^2).

Digression. An alternative method of expressing henry in terms of the fundamental MKSA units is by way of the relation $\Phi = LI$ between the magnetic flux Φ (whose units are weber), the inductance L (whose units are henry), and the electric current I (whose units are ampere). Thus, henry = weber/ampere = kg · m^2/(ampere^2 · sec^2).

One may also invoke the formula $\mathcal{E} = ½\mu_0 H^2$ for the energy-density of the H-field, where the units of $\mathcal{E}$ are joule/m^3. Thus, kg · m^2/(sec^2 · m^3) = (henry/m) · (ampere/m)^2, which yields henry = kg · m^2/(ampere^2 · sec^2).

On the right-hand side of the integral form of Maxwell's 3$^{\text{rd}}$ equation, the units of $\boldsymbol{B}$ (i.e., weber/m^2) must be multiplied into the units of the differential surface area $\mathrm{d}\boldsymbol{s}$ (i.e., m^2) to yield weber for the overall units of the integral. Differentiation with respect to time makes the units on the right-hand side of the equation equal to weber/sec = kg · m^2/(ampere · sec^3), which agree with the units on the left-hand side of the equation, namely, volt = kg · m^2/(ampere · sec^3).

iv) $\oint_{\text{closed surface}} \boldsymbol{B}(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{s} = 0$, obtained from $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0$ via Gauss's theorem.

The units of $\boldsymbol{B}$ are weber/m^2. Upon integration over a closed surface S, the units of $\boldsymbol{B}$ are multiplied by those of the infinitesimal surface area $\mathrm{d}\boldsymbol{s}$, which has the units of area (i.e., m^2). Thus, the units on the left-hand side of the integral form of Maxwell's 4$^{\text{th}}$ equation are weber.

Problem 4) a) The infinitesimal dipole moment formed between the positive and negative charges at y and $-y$ is $L\delta\rho(y)\mathrm{d}y$ times the separation $2y$ between these charges. The direction of this dipole is along $\hat{\boldsymbol{y}}$. Integrating over y (from $y = 0$ to L) yields

$$\boldsymbol{p} = \left[\int_0^L 2yL\delta\rho(y)\mathrm{d}y\right]\hat{\boldsymbol{y}} = 2L\delta\left[\int_0^L y\rho(y)\mathrm{d}y\right]\hat{\boldsymbol{y}}. \tag{1}$$

b) For sufficiently small L, the polarization of the pair of squares is given by their dipole moment $\boldsymbol{p}$ divided by the overall volume $2L^2\delta$ of the pair; that is,

$$\boldsymbol{P} = \boldsymbol{p}/(2L^2\delta) = L^{-1}\left[\int_0^L y\rho(y)\mathrm{d}y\right]\hat{\boldsymbol{y}}. \tag{2}$$

c) The magnitude of $\boldsymbol{P}(t)$ is the same as that of $\boldsymbol{P}$ in Eq.(2), but its direction is now given by $\cos(\omega t)\,\hat{\boldsymbol{y}} - \sin(\omega t)\,\hat{\boldsymbol{x}}$. Therefore,

$$\boldsymbol{P}(t) = L^{-1}\left[\int_0^L y\rho(y)\mathrm{d}y\right][\cos(\omega t)\,\hat{\boldsymbol{y}} - \sin(\omega t)\,\hat{\boldsymbol{x}}]. \tag{3}$$

d) As shown in figure (𝕓), at a point (x, y, z, t) on the surface of the red square at a distance $r = \sqrt{x^2 + y^2}$ from the z-axis, the velocity vector will be

$$\boldsymbol{v}(t) = r\omega[-\sin(\omega t + 90°)\,\hat{\boldsymbol{x}} + \cos(\omega t + 90°)\,\hat{\boldsymbol{y}}] = -r\omega[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}]. \tag{4}$$

The current-density at this point is, therefore, $\boldsymbol{J}(t) = \rho(r)\boldsymbol{v}(t)$. As for the blue square, the velocity at the corresponding point (i.e., at a distance r from the z-axis at time t) is the negative of $\boldsymbol{v}(t)$ of Eq.(4); however, the charge-density is also the negative of that at the corresponding point on the red square. Consequently, the current-density will be the same on the faces of both squares. Thus, at a distance r from the z-axis at time t, the current-density on both squares is

$$\boldsymbol{J}(t) = \rho(r)\boldsymbol{v}(t) = -r\omega\rho(r)[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}]. \tag{5}$$

e) The direction of the $\boldsymbol{J}$ vector at time t is the same as that of the velocity vector $\boldsymbol{v}(t)$ given by Eq.(4); that is, $-\cos(\omega t)\,\hat{\boldsymbol{x}} - \sin(\omega t)\,\hat{\boldsymbol{y}}$. The current passing through the surface of the red square along the direction of $\boldsymbol{J}(t)$ is $I_{\text{red}}(t) = \int_{\text{red square}} \boldsymbol{J}(t) \cdot \mathrm{d}\boldsymbol{s} = \int_{r=0}^{L} r\omega\rho(r)L\mathrm{d}r$. Doubling this current then yields the current passing through both (red and blue) squares, as follows:

$$I(t) = I_{\text{red}}(t) + I_{\text{blue}}(t) = 2L\omega\int_{r=0}^{L} r\rho(r)\mathrm{d}r. \tag{6}$$

f) For sufficiently small L, we divide the total current $I(t)$ of Eq.(6) by the surface area $2L^2$ of the pair to obtain the magnitude $\bar{J}(t)$ of the average current-density. Multiplication by the unit-vector along the flow direction then yields

$$\bar{\boldsymbol{J}}(t) = \frac{I(t)}{2L^2}[-\cos(\omega t)\,\hat{\boldsymbol{x}} - \sin(\omega t)\,\hat{\boldsymbol{y}}] = -(\omega/L)\left[\int_{r=0}^{L} r\rho(r)\mathrm{d}r\right][\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}]. \tag{7}$$

g) Comparing Eqs.(3) and (7), we finally arrive at the desired identity, namely,

$$\boldsymbol{J}_{\text{bound}}^{(e)} = \mathrm{d}\boldsymbol{P}(t)/\mathrm{d}t = -(\omega/L)\left[\int_{y=0}^{L} y\rho(y)\mathrm{d}y\right][\sin(\omega t)\,\hat{\boldsymbol{y}} + \cos(\omega t)\,\hat{\boldsymbol{x}}] = \bar{\boldsymbol{J}}(t). \tag{8}$$
