Opti 501 **2^{nd} Midterm Exam** (10/28/2021) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

1 pt **Problem 1**) a) Using direct integration, show that the Fourier transform of the function $\text{Rect}(t)$ is $\text{sinc}(\omega/2\pi)$. [By definition, $\text{sinc}(x) = \sin(\pi x)/(\pi x)$.]

2 pts b) Prove the scaling theorem of Fourier transformation, namely, $\mathcal{F}\{f(t/\alpha)\} = |\alpha|F(\alpha\omega)$, where α is a real-valued constant, and $F(\omega)$ is the Fourier transform of $f(t)$.

1 pt c) Let $T_1 = T_0 + \Delta T$ and $T_2 = T_0 - \Delta T$, where $0 < \Delta T \ll T_0$. Use the scaling theorem proved in (b) to derive the Fourier transform of $g(t) = [\text{Rect}(t/T_1) - \text{Rect}(t/T_2)]/\Delta T$.

1 pt d) Plot the function $g(t)$, then write an expression for $g_0(t) = lim_{\Delta T \to 0}\, g(t)$ in terms of the Dirac δ-function.

1 pt e) Using the result obtained in (c), find the Fourier transform of $g_0(t)$ as specified in (d).

Hint: $\sin(x) - \sin(y) = 2\sin[(x-y)/2]\cos[(x+y)/2]$; also, $\sin x \cong x$ when $|x| \ll 1$.

Problem 2) A thin spherical shell of radius R has a uniform (electric) surface-charge-density σ_0.

1 pt a) Express the charge-density $\rho(\boldsymbol{r}, t)$ in terms of σ_0 and a δ-function in spherical coordinates. Explain how the units on the two sides of the equation are consistent with each other.

2 pts b) Find the four-dimensional Fourier transform $\rho(\boldsymbol{k}, \omega)$ of the charge-density $\rho(\boldsymbol{r}, t)$.

1 pt c) Write down an expression for the scalar potential $\psi(\boldsymbol{k}, \omega)$ in terms of $\rho(\boldsymbol{k}, \omega)$ of part (b).

2 pts d) Compute the inverse Fourier transform of $\psi(\boldsymbol{k}, \omega)$ to arrive at the scalar potential $\psi(\boldsymbol{r}, t)$.

1 pt e) Use $\psi(\boldsymbol{r}, t)$ of part (d) to compute the E-field distribution both inside and outside the shell.

1 pt f) Confirm that the E-field discontinuity at the shell surface complies with the relevant Maxwell boundary condition.

Hint: $\int_0^\infty x^{-2}\sin(ax)\sin(bx)\,\mathrm{d}x = \begin{cases} \pi a/2; & b \ge a > 0, \\ \pi b/2; & a \ge b > 0. \end{cases}$ (Gradshteyn & Ryzhik **3.741**-3)

$$\boldsymbol{\nabla}\psi(r,\theta,\varphi) = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{\boldsymbol{\varphi}}.$$

Problem 3) Electromagnetic (EM) waves can exist in free space even in the absence of material media, which are the usual sources of these fields. In this problem, you are asked to explore the EM waves in free space under the conditions that $\rho_{\text{free}}(\boldsymbol{r}, t) = 0$, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) = 0$, $\boldsymbol{P}(\boldsymbol{r}, t) = 0$, and $\boldsymbol{M}(\boldsymbol{r}, t) = 0$.

1 pt a) Write Maxwell's equations for the EM fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ in the absence of all four sources.

1 pt b) Define the scalar potential $\psi(\boldsymbol{r}, t)$ and vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ in the usual way. Explain how these potentials manage to automatically satisfy Maxwell's third and fourth equations.

⋯ continued on page 2

1 pt c) Invoke Maxwell's first and second equations to arrive at two (coupled) partial differential equations that relate the previously defined potentials $\psi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ to each other.

1 pt d) Introduce the idea of the Lorenz gauge as an imposed relation between $\boldsymbol{A}(\boldsymbol{r},t)$ and $\psi(\boldsymbol{r},t)$ that would help to decouple the first and second equations of Maxwell arrived at in part (c). In this way, you should find two separate equations, one for the scalar potential, the other for the vector potential. (**Note**: Up until now, you have been working in the spacetime domain; therefore, your Lorenz gauge equation as well as your two separate equations for $\psi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ must contain the differential operators $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$, and $\partial/\partial t$.)

1 pt e) Moving to the Fourier domain, convert the equations obtained in part (d) to their corresponding equations for plane-waves $\psi(\boldsymbol{r},t) = \psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$ and $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$. In what follows, you may assume that both $\boldsymbol{k}$ and ω are real-valued.

1 pt f) Use the Fourier domain version of the Lorenz gauge equation to explain how this gauge takes advantage of the freedom to specify $\boldsymbol{A}_{\parallel}(\boldsymbol{k},\omega)$ without violating the original purpose of the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$, namely, a vector field whose curl must produce the B-field $\boldsymbol{B}(\boldsymbol{r},t)$.

1 pt g) Show that the Fourier domain equations for $\psi(\boldsymbol{k},\omega)$ and $\boldsymbol{A}(\boldsymbol{k},\omega)$ obtained in part (e) will have nonzero solutions for the corresponding plane-wave amplitudes only when the magnitude $|\boldsymbol{k}| = k$ of the k-vector happens to equal ω/c. (**Note**: the condition $k = \omega/c$ is a defining characteristic of EM plane-waves residing in free space.)

1 pt h) Use the Fourier domain version of the equation $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)$ to show that the plane-wave amplitude $\boldsymbol{B}(\boldsymbol{k},\omega)$ of the B-field is perpendicular to $\boldsymbol{k}$.

1 pt i) Use the Fourier domain version of the equation $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t$ in conjunction with the Lorenz gauge equation to show that the plane-wave amplitude $\boldsymbol{E}(\boldsymbol{k},\omega)$ of the E-field is perpendicular to both $\boldsymbol{k}$ and $\boldsymbol{B}(\boldsymbol{k},\omega)$.

1 pt j) Using the results of part (i), argue that $\boldsymbol{B}(\boldsymbol{k},\omega)$ is related to $\boldsymbol{E}(\boldsymbol{k},\omega)$ through a 90° rotation around the k-vector, followed by division by the speed c of light in vacuum.

1 pt k) Invoking the fact that $\boldsymbol{M}(\boldsymbol{r},t) = 0$ in free space, use the result obtained in part (j) to show that $\boldsymbol{H}(\boldsymbol{k},\omega)$ is related to $\boldsymbol{E}(\boldsymbol{k},\omega)$ through a 90° rotation around the k-vector, followed by division by the impedance Z_0 of free space.

Hint: The vector identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ could be useful.
