Opti 501 **2^{nd} Midterm Exam** (11/5/2020) **Time: 75 minutes**

Please write your name and ID number on the first page before scanning/photographing the pages. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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2 pts **Problem 1**) a) The static electric point-dipole $p_0\hat{\boldsymbol{z}}$ sits at the origin of coordinates at $(x, y, z) = (0, 0, 0)$. Write expressions for the corresponding polarization $\boldsymbol{P}(\boldsymbol{r}, t)$ and the bound electric charge-density $\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r}, t)$. Using words and graphs, explain how your mathematical formula for the bound charge-density represents the physical structure of the electric dipole.

2 pts b) The static magnetic point-dipole $m_0\hat{\boldsymbol{z}}$ sits at the origin of coordinates at $(x, y, z) = (0, 0, 0)$. Write expressions for the corresponding magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$ and the bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{r}, t)$. Using words and graphs, explain how your mathematical formula for the bound current-density represents the physical structure of the magnetic dipole.

2 pts c) Find the four-dimensional Fourier transforms of $\rho_{\text{bound}}^{(\text{e})}$ and $\boldsymbol{J}_{\text{bound}}^{(\text{e})}$ obtained in parts (a) and (b).

Hint: $\boldsymbol{\nabla} \cdot \boldsymbol{V} = \partial V_x/\partial x + \partial V_y/\partial y + \partial V_z/\partial z.$

$\boldsymbol{\nabla} \times \boldsymbol{V} = (\partial V_z/\partial y - \partial V_y/\partial z)\hat{\boldsymbol{x}} + (\partial V_x/\partial z - \partial V_z/\partial x)\hat{\boldsymbol{y}} + (\partial V_y/\partial x - \partial V_x/\partial y)\hat{\boldsymbol{z}}.$

Problem 2) The radiated $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of an infinitely long, thin, straight wire carrying the electric current $I_0 \cos(\omega_0 t)$ along the z-axis are found in the cylindrical coordinates (ρ, φ, z) to be

$$\boldsymbol{E}(\boldsymbol{r}, t) = -(\mu_0 I_0 \omega_0/4)[J_0(\rho\omega_0/c)\cos(\omega_0 t) + Y_0(\rho\omega_0/c)\sin(\omega_0 t)]\hat{\boldsymbol{z}},$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = (I_0\omega_0/4c)[J_1(\rho\omega_0/c)\sin(\omega_0 t) - Y_1(\rho\omega_0/c)\cos(\omega_0 t)]\hat{\boldsymbol{\varphi}}.$$

It is also found that a constant current I_0 along the same wire produces no electric field but a time-independent magnetic field $\boldsymbol{H}(\boldsymbol{r}) = (I_0/2\pi\rho)\hat{\boldsymbol{\varphi}}$.

3 pts a) Show that the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the constant current can be derived as a limiting case of the fields produced by the oscillating current.

3 pts b) Show that the far field radiated by the oscillating current has the correct retarded term $\cos[\omega_0(t - \rho/c) + \varphi_0]$, where φ_0 is a constant phase. Verify that the Poynting vector in the far field, while aligned with $\hat{\boldsymbol{\rho}}$, declines in inverse proportion to the distance ρ from the wire.

Hint: The small-argument limiting forms of the Bessel functions when $x \to 0$ are known to be

$J_n(x) \sim \frac{(x/2)^n}{n!}$; $\quad n \geq 0,$

$Y_0(x) \sim \frac{2}{\pi}[\mathcal{C} + \ln(x/2)]$; $\quad$ Euler constant $\mathcal{C} = 0.577215\cdots,$

$Y_n(x) \sim -\frac{(n-1)!}{\pi}(x/2)^{-n}$; $\quad n \geq 1.$

The large-argument limiting forms of the Bessel functions when $x \to \infty$ are given by

$$J_n(x) \sim \sqrt{2/(\pi x)}\cos[x - (n\pi/2) - (\pi/4)],$$

$$Y_n(x) \sim \sqrt{2/(\pi x)}\sin[x - (n\pi/2) - (\pi/4)].$$

Other useful identities are: $\cos x \cos y + \sin x \sin y = \cos(x - y)$ and $\sin x \cos y \pm \cos x \sin y = \sin(x \pm y)$. Also recall that $\sin(x \pm \tfrac{1}{2}\pi) = \pm \cos x$.

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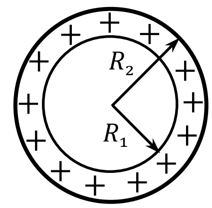
4 pts **Problem 3**) Show that the Lorenz gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{r},t)+\frac{\partial\psi(\boldsymbol{r},t)}{c^2\partial t}=0$ is a direct consequence of the charge-current continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r},t)+\frac{\partial\rho(\boldsymbol{r},t)}{\partial t}=0$.

Problem 4) An electrically-charged spherical shell has inner radius R_1, outer radius R_2, and a uniform, time-independent charge-density ρ_0.



2 pts a) Find the 4-dimensional Fourier transform of $\rho(\boldsymbol{r},t)$.

1 pt b) Write an expression for the scalar potential $\psi(\boldsymbol{k},\omega)$.

2 pts c) Find the inverse Fourier transform $\psi(\boldsymbol{r},t)$ of $\psi(\boldsymbol{k},\omega)$.

2 pts d) Find the E-field inside the hollow cavity ($r\le R_1$), within the wall of the shell ($R_1\le r\le R_2$), and outside the sphere ($r\ge R_2$).

1 pt e) Calculate the average E-field within the wall of the shell ($R_1\le r\le R_2$).

1 pt f) For a sufficiently thin shell (i.e., when $R_1\to R_2$), show that the average E-field within the wall of the shell approaches one-half the E-field immediately outside the shell (i.e., at $r=R_2^+$).

Hint: $\int_0^\pi \sin\varphi\, e^{\pm i\alpha\cos\varphi}\mathrm{d}\varphi=(2/\alpha)\sin\alpha$; $\int x\sin(\beta x)\,\mathrm{d}x=[\sin(\beta x)-\beta x\cos(\beta x)]/\beta^2$;

$$\int_0^\infty \frac{[\sin(kR)-kR\cos(kR)]\sin(kr)}{k^4}\mathrm{d}k=\begin{cases}\pi R^3/6\,; & r\ge R,\\ \pi r(3R^2-r^2)/12\,; & r\le R.\end{cases}$$
