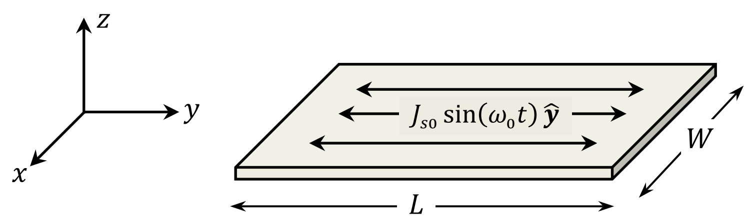
Opti 501 **2nd Midterm Exam** (11/7/2019) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) A thin, rectangular electrical conductor of length L, width W, and negligible thickness carries the surface current density $J_{s0}\sin(\omega_0 t)\,\hat{\boldsymbol{y}}$ along the direction of its length, as shown in the figure. The conductor as a whole is electrically neutral, that is, its net total electrical charge is zero.



1 pt a) Use special functions Rect(·) and $\delta(\cdot)$ to write an expression for the current-density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$.

2 pts b) Use the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J}+\partial\rho/\partial t=0$, to determine the electric charge-density $\rho_{\text{free}}(\boldsymbol{r},t)$ throughout the conductor.

1 pt c) Explain in words where the charges appear on the conductor and how they vary with time.

Problem 2) Inside an infinitely tall, hollow, perfectly electrically conducting cylinder of inner radius R, the scalar and vector potentials in the (r,φ,z) cylindrical coordinate system are specified as $\psi(\boldsymbol{r},t)=0$ and $\boldsymbol{A}(\boldsymbol{r},t)=A_0 J_0(\omega_0 r/c)\cos(\omega_0 t)\,\hat{\boldsymbol{z}}$. Here $J_0(\cdot)$ is a Bessel function of the first kind, 0th order, ω_0 is the constant oscillation frequency, A_0 is an arbitrary real-valued constant, and $\hat{\boldsymbol{z}}$ is along the axis of the cylinder. The frequency ω_0 is chosen such that the radius R of the cylindrical cavity is at a zero of the Bessel function, that is, $J_0(\omega_0 R/c)=0$.

3 pts a) Find the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$ inside the cylindrical cavity.

2 pts b) What is the surface current-density $\boldsymbol{J}_s(t)$ induced on the interior wall of the hollow cylinder?

Hint: $\boldsymbol{E}=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t$; $\mu_0\boldsymbol{H}=\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}$; $J_0'(x)=-J_1(x)$; also see Table on page 2.

Problem 3) Let the Fourier transform of a free current-density throughout space and time be

$$\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)=I_0\delta(k-k_0)[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]\hat{\boldsymbol{k}}.$$

In the above equation, $\boldsymbol{k}=k\hat{\boldsymbol{k}}$ is the k-vector, k_0 [1/meter] is a constant wave-number, ω_0 [1/sec] is a constant frequency, and I_0 [ampere] is a constant current amplitude.

3 pts a) What is the current-density distribution $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ in the spacetime domain?

3 pts b) Invoking the charge-current continuity equation in the Fourier domain, $\omega\rho(\boldsymbol{k},\omega)=\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{k},\omega)$, determine $\rho_{\text{free}}(\boldsymbol{k},\omega)$, then carry out an inverse Fourier transformation to find $\rho_{\text{free}}(\boldsymbol{r},t)$.

2 pts c) Confirm that the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\partial\rho_{\text{free}}(\boldsymbol{r},t)/\partial t=0$, is satisfied in the spacetime domain. (See Table on page 2.)

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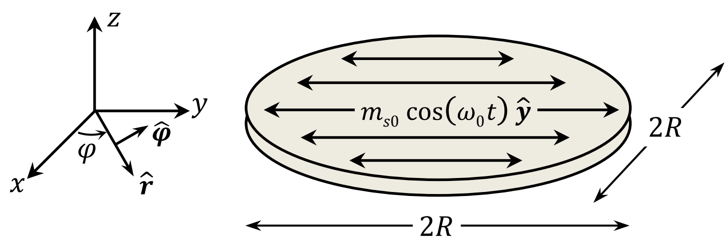
Hint: $\int_0^{\pi} \sin\varphi \cos\varphi \exp(\mathrm{i}\beta\cos\varphi)\,\mathrm{d}\varphi = \frac{2\mathrm{i}\,(\sin\beta - \beta\cos\beta)}{\beta^2}$; $\int_0^{\pi} \sin\varphi \exp(\mathrm{i}\beta\cos\varphi)\,\mathrm{d}\varphi = \frac{2\sin\beta}{\beta}$.

Problem 4) The figure below shows a thin, circular disk of radius R and negligible thickness, whose magnetic dipole moment density (per unit-area) is specified as $m_{s0}\cos(\omega_0 t)\,\hat{\boldsymbol{y}}$.



2 pts a) Using the special functions Circ(·) and $\delta(\cdot)$, write an expression for the magnetization distribution $\boldsymbol{M}(\boldsymbol{r},t)$.

2 pts b) What are the bound magnetic charge-density $\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t)$ and magnetic current-density $\boldsymbol{J}_{\text{bound}}^{(m)}(\boldsymbol{r},t)$ of the disk?

2 pts c) What are the disk's bound electric charge and current densities, $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)$?

2 pts d) Explain in words where the charges and currents are located and how they vary with time.

Hint: $\hat{\boldsymbol{x}} = \cos\varphi\,\hat{\boldsymbol{r}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{y}} = \sin\varphi\,\hat{\boldsymbol{r}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$.

Gradient, Divergence, and Curl

In Cartesian coordinates:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z},$$

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{\boldsymbol{z}}.$$

In cylindrical coordinates:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{\partial f}{\rho\partial\varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial(\rho V_\rho)}{\rho\partial\rho} + \frac{\partial V_\varphi}{\rho\partial\varphi} + \frac{\partial V_z}{\partial z},$$

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \left(\frac{\partial V_z}{\rho\partial\varphi} - \frac{\partial V_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left[\frac{\partial(\rho V_\varphi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\varphi}\right]\hat{\boldsymbol{z}}.$$

In spherical coordinates:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial f}{r\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\boldsymbol{\varphi}},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial(r^2 V_r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta V_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi},$$

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta V_\varphi)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right]\hat{\boldsymbol{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial(rV_\varphi)}{r\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}.$$