Opti 501 **2^{nd} Midterm Exam** (10/25/2018) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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2 pts **Problem 1**) a) Write Maxwell's equations in their standard form in the Fourier domain. All the sources and all the fields must be expressed as functions of $(\boldsymbol{k}, \omega)$.

2 pts b) Eliminate the fields $\boldsymbol{D}(\boldsymbol{k}, \omega)$ and $\boldsymbol{B}(\boldsymbol{k}, \omega)$ from the equations, then solve the equations for $\boldsymbol{E}(\boldsymbol{k}, \omega)$ and $\boldsymbol{H}(\boldsymbol{k}, \omega)$ in terms of the sources.

2 pts c) Identify the contributions of the various sources to the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields. For instance, identify the mathematical expression in which the free charge density appears in the formulas for the $\boldsymbol{E}$ field and/or the $\boldsymbol{H}$ field, and the expression for the bound magnetic current-density, etc.

Hint: The vector identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ will be useful.

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2 pts **Problem 2**) a) Find the Fourier transform of the 2-dimensional function $f(x, y) = \text{circ}(r)$, where $\text{circ}(r)$ equals 1.0 when $r < 1$, and 0.0 when $r > 1$. Here $r = \sqrt{x^2 + y^2}$ is the radial coordinate in a 2-dimensional polar coordinate system.

Hint: The Fourier transform turns out to be a function of the magnitude k of the 2-dimensional k-vector, namely, $|\boldsymbol{k}| = |k_x \hat{\boldsymbol{x}} + k_y \hat{\boldsymbol{y}}|$. The Fourier kernel $\boldsymbol{k} \cdot \boldsymbol{r}$ in this 2-dimensional problem is initially written as $k_x x + k_y y$.

$$\int_0^{2\pi} \exp(\pm \mathrm{i}\beta \cos\varphi)\, \mathrm{d}\varphi = 2\pi J_0(\beta); \qquad \int x J_0(x) \mathrm{d}x = x J_1(x).$$

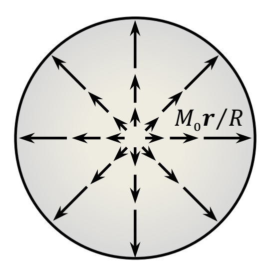
$J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind, orders 0 and 1, respectively.

2 pts b) Find the Fourier transform of the scaled function $\alpha^{-2} f(x/\alpha, y/\alpha) = \alpha^{-2}\text{circ}(r/\alpha)$, where α is a positive real constant.

2 pts c) In the limit when $\alpha \to 0$, the function $\alpha^{-2}\text{circ}(r/\alpha)$ approaches a 2-dimensional δ-function that is equivalent to $\pi\delta(x)\delta(y)$. What is the (2-dimensional) Fourier transform of this δ-function obtained in part (b) in the limit when $\alpha \to 0$?

Hint: The Taylor series expansion $J_1(x) = \frac{1}{2}x - \frac{(x/2)^3}{1!2!} + \frac{(x/2)^5}{2!3!} - \frac{(x/2)^7}{3!4!} + \cdots$ indicates that, for small x, $J_1(x) \to \frac{1}{2}x$.

Problem 3) The solid sphere of radius R depicted on the right-hand side is filled with a permanently magnetized material whose magnetization distribution is specified as $\boldsymbol{M}(\boldsymbol{r}, t) = M_0 \boldsymbol{r}/R$.



3 pts a) Find the Fourier transform $\boldsymbol{M}(\boldsymbol{k}, \omega)$ of the magnetization distribution $\boldsymbol{M}(\boldsymbol{r}, t)$.

Hint: $\int_0^{\pi} \sin\theta \cos\theta \exp(\pm \mathrm{i}\zeta \cos\theta)\, \mathrm{d}\theta = \pm 2\mathrm{i}\,[\sin(\zeta) - \zeta\cos(\zeta)]/\zeta^2$.

$\int_0^{\zeta_0} (\zeta \sin\zeta - \zeta^2 \cos\zeta) \mathrm{d}\zeta = 3\sin\zeta_0 - 3\zeta_0 \cos\zeta_0 - \zeta_0^2 \sin\zeta_0$.

2 pts b) Considering that the bound electric current-density associated with magnetization is given by $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r}, t) = \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}(\boldsymbol{r}, t)$, determine the Fourier transform $\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{k}, \omega)$ of the bound current-density inside the sphere.

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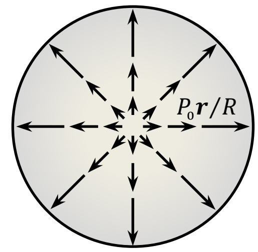
1 pt c) Show that the magnetic B-field, $\boldsymbol{B}(\boldsymbol{r},t)$, is zero everywhere (i.e., inside as well as outside the sphere.)

1 pt d) Find the magnetic H-field, $\boldsymbol{H}(\boldsymbol{r},t)$, inside the sphere.

Problem 4) The solid sphere of radius R depicted on the right-hand side is filled with a permanently polarized material whose polarization distribution is specified as $\boldsymbol{P}(\boldsymbol{r},t) = P_0\boldsymbol{r}/R$.



1 pt a) Find the Fourier transform $\boldsymbol{P}(\boldsymbol{k},\omega)$ of the polarization distribution $\boldsymbol{P}(\boldsymbol{r},t)$.

Hint: If you have already solved Problem 3(a), you may use the same result here, albeit with P_0 substituted for M_0. If not, proceed to calculate the Fourier transform with the aid of the following integrals, then use the result in Problem 3(a) — after substituting M_0 for P_0.

$$\int_0^\pi \sin\theta\cos\theta\exp(\pm \mathrm{i}\zeta\cos\theta)\,\mathrm{d}\theta = \pm 2\mathrm{i}\,[\sin(\zeta) - \zeta\cos(\zeta)]/\zeta^2.$$

$$\int_0^{\zeta_0}(\zeta\sin\zeta - \zeta^2\cos\zeta)\mathrm{d}\zeta = 3\sin\zeta_0 - 3\zeta_0\cos\zeta_0 - \zeta_0^2\,\sin\zeta_0.$$

1 pt b) Considering that the bound electric charge-density associated with polarization is given by $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$, determine the Fourier transform $\rho_{\text{bound}}^{(e)}(\boldsymbol{k},\omega)$ of the bound charge-density inside the sphere.

2 pts c) In terms of P_0, R, $\boldsymbol{k}$, ω, and ε_0, write an expression for the Fourier transform $\boldsymbol{E}(\boldsymbol{k},\omega)$ of the E-field distribution throughout the entire spacetime.

2 pts d) Use inverse Fourier transformation to find the E-field $\boldsymbol{E}(\boldsymbol{r},t)$ in the spacetime domain.

Hint:
$$\int_{k=0}^{\infty}\frac{[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)][\sin(kr)-kr\cos(kr)]}{k^4}\mathrm{d}k = \begin{cases} -\pi r^3/2\,; & r<R, \\ -\pi r^3/4\,; & r=R, \\ 0; & r>R. \end{cases}$$
