Opti 501 $\quad$ 2nd Midterm Exam (10/26/2017) $\quad$ Time: 75 minutes

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

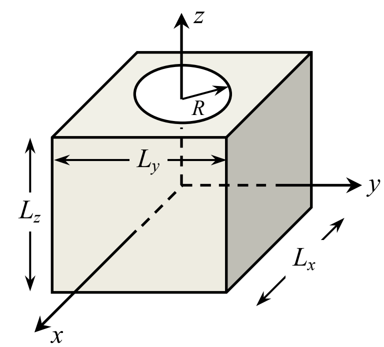
Note: Bold symbols represent vectors and vector fields.

---

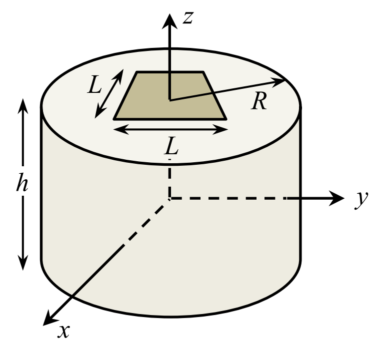
5 pts **Problem 1**) The sifting property of Dirac's delta-function acting on an arbitrary function $f(x)$ is generally expressed as $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\mathrm{d}x = f(x_0)$, assuming that $f(x)$ is continuous at $x = x_0$. Use the method of *integration-by-parts* to arrive at a similar expression for the sifting property of the derivative $\delta'(x)$ of the delta-function.

---

5 pts **Problem 2**) a) A charge-density distribution is described as having a constant value ρ_0 inside a rectangular cuboid of dimensions $L_x \times L_y \times L_z$, except within a right-circular cylindrical hole of radius R at the center, where the charge-density is zero (see the figure). Write the mathematical expression of the charge-density distribution $\rho(\boldsymbol{r},t)$ throughout the entire spacetime using the special functions Rect(·) and Circ(·).



5 pts b) A magnetization distribution is described as having a time-dependent magnitude $M_0 \cos(\omega_0 t)$ within a solid cylinder of radius R and height h. Inside a rectangular cuboid of dimensions $L \times L \times h$ centered on the cylinder axis, the magnetization at $t = 0$ points along the positive z-axis, whereas outside the cuboid the magnetization at $t = 0$ points along the negative z-axis (see the figure). Using the special functions Circ(·) and Rect(·), write the mathematical expression for the magnetization $\boldsymbol{M}(\boldsymbol{r},t)$ throughout the entire spacetime.



Problem 3) In the xyz coordinate system, the current-density distribution is given by

$$\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}_1 \cos(\boldsymbol{k}_0 \cdot \boldsymbol{r}) \cos(\omega_0 t) + \boldsymbol{J}_2 \sin(\boldsymbol{k}_0 \cdot \boldsymbol{r}) \sin(\omega_0 t).$$

Here ω_0 is a constant real-valued frequency, while $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{k}_0$ are constant real-valued vectors.

2 pts a) Find the Fourier transform $\boldsymbol{J}(\boldsymbol{k},\omega)$ of the current-density distribution $\boldsymbol{J}(\boldsymbol{r},t)$.

3 pts b) Use the charge-current continuity equation to determine the charge-density distribution $\rho(\boldsymbol{r},t)$ and also its Fourier transform $\rho(\boldsymbol{k},\omega)$.

2 pts c) Working in the Lorenz gauge, find the scalar and vector potentials $\psi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ associated with the above charge and current densities.

3 pts d) Find the electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ produced by the above charge and current densities.

Hint: $\cos x = [\exp(\mathrm{i}x) + \exp(-\mathrm{i}x)]/2$ and $\sin x = [\exp(\mathrm{i}x) - \exp(-\mathrm{i}x)]/(2\mathrm{i})$.
The 4D Fourier transform of the function $\exp[\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)]$ is $(2\pi)^4 \delta(\boldsymbol{k} - \boldsymbol{k}_0)\delta(\omega - \omega_0)$.