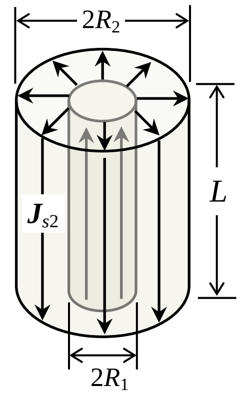
Opti 501 **2^{nd} Midterm Exam** (11/10/2015) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) The figure shows two concentric hollow cylinders of length L carrying a constant electric current in opposite directions. The inner cylinder's radius and surface-current-density are R_1 and $J_{s1}\hat{\boldsymbol{z}}$, while the corresponding parameters for the outer cylinder are R_2 and $J_{s2}\hat{\boldsymbol{z}}$. The end-caps attached to the top and bottom of the cylinders close the current path, so that $\boldsymbol{\nabla}\cdot\boldsymbol{J}=0$ everywhere.



3 Pts a) Express the surface current-density in the upper and lower end-caps, and also that in the outer cylinder, in terms of the surface current-density J_{s1} of the inner cylinder.

4 Pts b) Determine the magnetic field distribution $\boldsymbol{H}(\boldsymbol{r},t)$ in the entire space (i.e., both inside and outside the cavity formed by the two cylinders and their end-caps).

Hint: You may guess the answer to part (b), then check that it satisfies Maxwell's equations as well as the relevant boundary conditions.

Problem 2) Vector spherical harmonics are special solutions of Maxwell's equations in spherical coordinates. The lowest-order vector spherical harmonic in free space has the following scalar and vector potentials:

$$\psi(\boldsymbol{r},t)=0,$$

$$\boldsymbol{A}(\boldsymbol{r},t)=A_0\left[\frac{\sin(k_0 r)}{(k_0 r)^2}-\frac{\cos(k_0 r)}{k_0 r}\right]\sin\theta\cos(\omega t)\,\hat{\boldsymbol{\varphi}}.$$

Here A_0 is a real-valued constant, r is the radial distance from the origin, θ is the polar coordinate, ω is the oscillation frequency, $k_0=\omega/c$ is the wave-number, and $\hat{\boldsymbol{\varphi}}$ is the unit-vector in the azimuthal direction.

2 Pts a) Show that the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ does *not* have a singularity at the origin of coordinates.

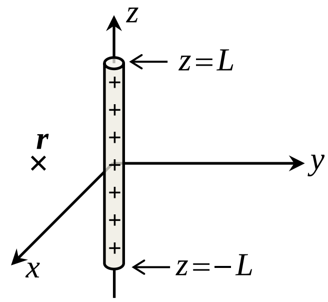
2 Pts b) Confirm that the vector spherical harmonic satisfies the Lorenz gauge condition.

1 Pt c) Find the E-field of the vector spherical harmonic throughout space and time.

2 Pts d) Find the B-field of the vector spherical harmonic throughout space and time.

2 Pts e) Compute the Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$ for the vector spherical harmonic, and explain how the corresponding electromagnetic energy travels in space-time.

Problem 3) A uniformly-charged rod of length $2L$ and negligible diameter resides on the z-axis between $z = -L$ and $z = L$, as shown. The linear charge-density of the rod is λ_0 coulomb/meter.



1 Pt a) Use some of the commonly used special functions to write an expression for the charge-density $\rho(\boldsymbol{r}, t)$.

2 Pts b) Write the direct integral for the scalar potential $\psi(\boldsymbol{r}, t)$ in terms of $\rho(\boldsymbol{r}, t)$, then evaluate the integral.

2 Pts c) Show that in the limit when $L \to \infty$, the result obtained in part (b) yields the expected E-field distribution around a long rod.

2 Pts d) Find the Fourier transform $\rho(\boldsymbol{k}, \omega)$ of $\rho(\boldsymbol{r}, t)$ derived in part (a).

2 Pts e) Write the expression for the scalar potential $\psi(\boldsymbol{k}, \omega)$, and proceed to compute its inverse Fourier transform $\psi(\boldsymbol{r}, t)$. Your final answer must agree with the result obtained in part (b).

Hint: $\int \frac{dz}{\sqrt{r^2+z^2}} = \ln\left(z + \sqrt{r^2 + z^2}\right).$ (G&R **2.261**)

$\int_0^{2\pi} \exp(\pm \mathrm{i}\beta \cos x) dx = 2\pi J_0(\beta).$ (G&R **3.915**-2)

$\int_0^{\infty} \frac{x J_0(ax)}{x^2+\beta^2} dx = K_0(a\beta); \quad a > 0, \quad \mathrm{Re}(\beta) > 0.$ (G&R **6.532**-4)

$\int_0^{\infty} x^{-1} \sin(ax)\, K_0(bx) dx = ½\pi \ln\left[(a/b) + \sqrt{1 + (a/b)^2}\right]; \quad a > 0, \; b > 0.$

(G&R **6.699**-3, **9.121**-28, **8.338**-2)