Opti 501 **2^{nd} Midterm Exam** (10/24/2013) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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5 Pts **Problem 1**) The function $f(t) = \cos(\omega_0 t)$ is defined over the entire time axis. The Fourier transform $F(\omega)$ of $f(t)$ is given by

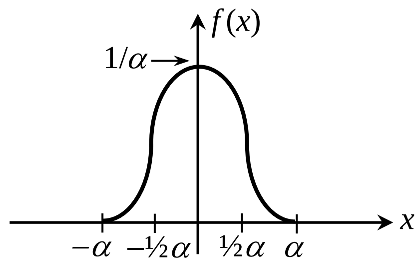
$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i\omega t)\,\mathrm{d}t.$$

Since $f(t)$ oscillates with undiminished amplitude as $t \to \pm\infty$, one must first force the function to approach zero in these limits, then reduce the effects of this artificial decay by allowing some adjustable parameter to go to zero. Use the above procedure to determine the function $F(\omega)$.

Hint: $\cos(\omega_0 t) = ½[\exp(i\omega_0 t) + \exp(-i\omega_0 t)]$.

Problem 2) Consider the function $f(x)$ defined on the entire x-axis, having nonzero values only over the interval $-\alpha < x < \alpha$, as follows:

$$f(x) = \begin{cases} 0, & x \le -\alpha; \\ (2/\alpha)[1 + (x/\alpha)]^2, & -\alpha \le x \le -½\alpha; \\ (2/\alpha)[½ - (x/\alpha)^2], & -½\alpha \le x \le ½\alpha; \\ (2/\alpha)[1 - (x/\alpha)]^2, & ½\alpha \le x \le \alpha; \\ 0, & x \ge \alpha. \end{cases}$$



2 Pts a) Find the area under $f(x)$, then show that $f(x) \to \delta(x)$ when $\alpha \to 0$.

2 Pts b) Plot the function $f'(x)$ and verify that $f'(x) \to \delta'(x)$ in the limit when $\alpha \to 0$ by confirming the corresponding sifting property.

2 Pts c) Plot the function $f''(x)$ and verify that $f''(x) \to \delta''(x)$ in the limit when $\alpha \to 0$ by confirming the corresponding sifting property.

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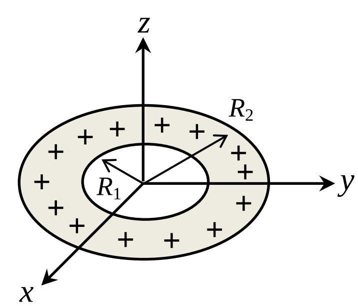
6 Pts **Problem 3**) A uniformly-charged circular ring having inner radius R_1, outer radius R_2, areal charge-density σ_0, and negligible thickness along the z-axis is located in the xy-plane and centered on the z-axis. Denoting the position vector in the xy-plane by $\boldsymbol{r}_\parallel = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ and working in the cylindrical coordinate system, the charge-density distribution may be written as follows:



$$\rho_{\text{free}}(r_\parallel, \phi, z, t) = \sigma_0[\text{Circ}(r_\parallel/R_2) - \text{Circ}(r_\parallel/R_1)]\delta(z).$$

Find the Fourier transform $\rho_{\text{free}}(\boldsymbol{k}, \omega)$ of the above charge distribution.

Hint: $\int_0^{2\pi} \exp(\pm i\beta\cos x)\,\mathrm{d}x = 2\pi J_0(\beta)$ and $\int x J_0(x) dx = x J_1(x)$, where $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind, order 0 and 1, respectively.

Problem 4) Maxwell's macroscopic equations may be written in terms of mixed (i.e., both electric and magnetic) bound-charge and bound-current densities as follows:

$$\varepsilon_\mathrm{o}\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = \rho_\mathrm{free}(\boldsymbol{r},t) - \boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t),$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_\mathrm{free}(\boldsymbol{r},t) + \frac{\partial\boldsymbol{P}(\boldsymbol{r},t)}{\partial t} + \varepsilon_\mathrm{o}\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t},$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial\boldsymbol{M}(\boldsymbol{r},t)}{\partial t} - \mu_\mathrm{o}\frac{\partial\boldsymbol{H}(\boldsymbol{r},t)}{\partial t},$$

$$\mu_\mathrm{o}\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t).$$

4 Pts a) Transform the above equations to the Fourier domain, where all sources and fields become functions of $(\boldsymbol{k},\omega)$ instead of $(\boldsymbol{r},t)$.

4 Pts b) Without introducing any potentials, directly solve the Fourier-domain equations obtained in part (a) to find the fields $\boldsymbol{E}(\boldsymbol{k},\omega)$ and $\boldsymbol{H}(\boldsymbol{k},\omega)$ in terms of the source distributions $\rho_\mathrm{free}(\boldsymbol{k},\omega)$, $\boldsymbol{J}_\mathrm{free}(\boldsymbol{k},\omega)$, $\boldsymbol{P}(\boldsymbol{k},\omega)$ and $\boldsymbol{M}(\boldsymbol{k},\omega)$.

Hint: In the Fourier domain, cross-multiply Maxwell's 2^{nd} and 3^{rd} equations into $\boldsymbol{k}$, then use the vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ to simplify the resulting equations.