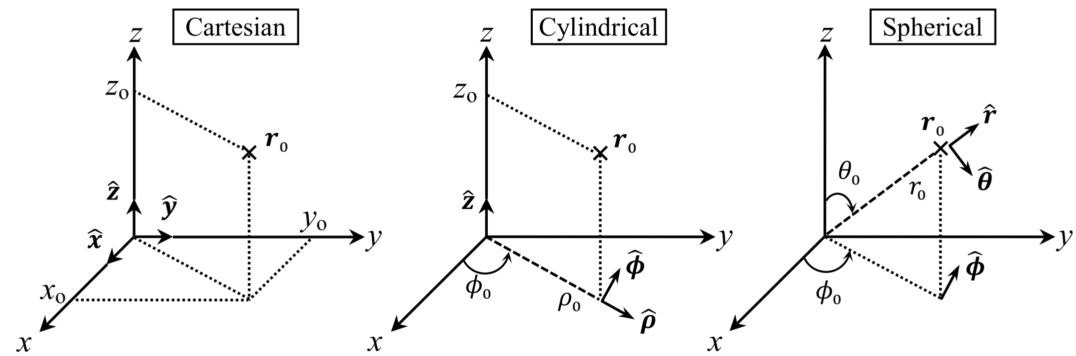
Opti 501 **1st Midterm Exam** (9/28/2021) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

6 pts **Problem 1**) In cylindrical and spherical coordinate systems, the unit-vector $\hat{\boldsymbol{\phi}}$ is related to the unit-vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ of the Cartesian coordinate system as follows: $\hat{\boldsymbol{\phi}} = -(\sin\phi)\hat{\boldsymbol{x}} + (\cos\phi)\hat{\boldsymbol{y}}$. In a similar way, relate $\hat{\boldsymbol{\rho}}$ of the cylindrical system to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. How are $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ of the spherical system related to $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$? Finally, noting that $\hat{\boldsymbol{z}}$ has no component along $\hat{\boldsymbol{\phi}}$, express $\hat{\boldsymbol{z}}$ in terms of the unit-vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ of the spherical coordinate system.

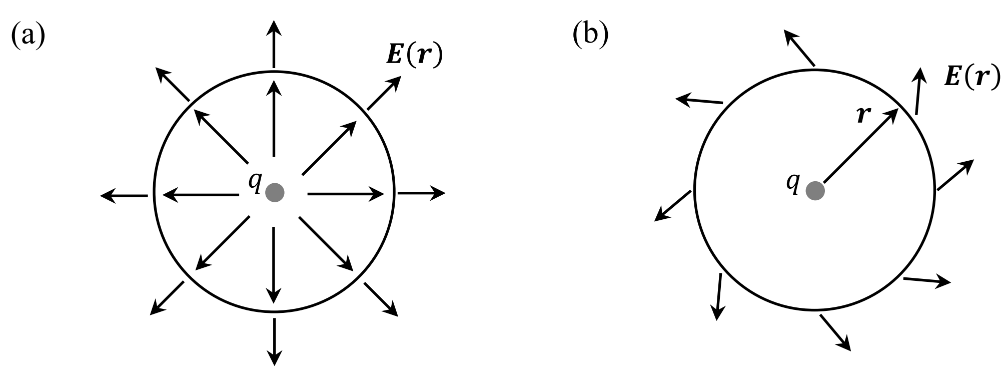


Problem 2) According to the Coulomb law of electrostatics, the E-field produced at a point $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ by a point-charge q sitting at the origin of coordinates is $\boldsymbol{E}(\boldsymbol{r}) = q\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2)$.

2 pts a) Using a sphere of radius r surrounding the point charge q in conjunction with Maxwell's 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$, derive the Coulomb law; see Fig.(a).

2 pts b) Use an argument based on the symmetry of space to show that the E-field on the surface of the sphere *cannot* be tilted away from the radial direction in the manner depicted in Fig.(b).

2 pts c) In this electrostatic system, invoke Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r}) = 0$, to argue that the azimuthal tilt of the E-field shown in Fig.(b) is also incompatible with Maxwell's 3rd equation.



5 pts **Problem 3**) A time-independent vector field $\boldsymbol{A}(\boldsymbol{r})$ is specified in a cylindrical coordinate system (ρ, ϕ, z), as follows:

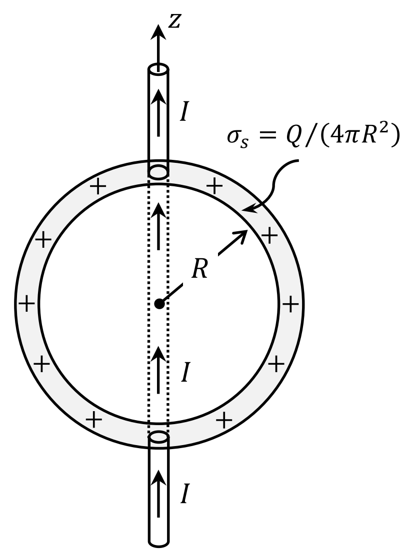
$$\boldsymbol{A}(\boldsymbol{r}) = \begin{cases} A_0 \rho \hat{\boldsymbol{\phi}}; & \rho \le R, \\ (A_0 R^2/\rho)\hat{\boldsymbol{\phi}}; & \rho \ge R. \end{cases}$$

Here, $R > 0$ and A_0 are arbitrary real-valued constants. Find the vector field $\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r})$ everywhere in space — that is, both inside and outside the infinitely-long cylinder of radius R.

Hint: The curl operator in the cylindrical (ρ, ϕ, z) coordinate system is given by

$$\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}) = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}}.$$

Problem 4) A thin, uniformly-charged, spherical shell of radius R and surface-charge-density $\sigma_s = Q/(4\pi R^2)$ is centered at the origin of the spherical (r, θ, ϕ) coordinate system. The E-field produced by this spherical charge distribution is $\boldsymbol{E}(\boldsymbol{r}) = Q\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2)$ outside the sphere, and zero inside. A thin, straight, infinitely-long wire carrying the constant current I passes through the poles of the sphere. The time-independent magnetic field $\boldsymbol{H}(\boldsymbol{r}) = (I/2\pi\rho)\hat{\boldsymbol{\phi}}$ produced by the current-carrying wire circulates around the z-axis. Here, $\rho = r\sin\theta$ is the distance between the point $\boldsymbol{r} = (r, \theta, \phi)$ and the z-axis.



3 pts a) What is the energy density $\mathcal{E}(\boldsymbol{r})$ of the electromagnetic field at each and every point $\boldsymbol{r}$?

3 pts b) Find the Poynting vector $\boldsymbol{S}(\boldsymbol{r})$ throughout the entire space, then show that $\boldsymbol{\nabla} \cdot \boldsymbol{S}(\boldsymbol{r}) = 0$ everywhere except on the z-axis in the two regions above and below the spherical shell.

2 pts c) Show that the Poynting vector $\boldsymbol{S}(\boldsymbol{r})$ found in part (b) is consistent with a picture of the electromagnetic energy in which the lower part of the current-carrying wire (i.e., below the sphere) emits, while the upper part (above the sphere) absorbs the electromagnetic energy.

Hint: In the spherical (r, θ, ϕ) coordinate system, $\boldsymbol{\nabla} \cdot \boldsymbol{S} = \frac{1}{r^2}\frac{\partial(r^2 S_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, S_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial S_\phi}{\partial\phi}$.
