Opti 501 **1st Midterm Exam** (10/1/2020) **Time: 75 minutes**

Please write your name and ID number on the first page before scanning/photographing the pages. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

5 pts **Problem 1**) The identity $\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$ holds for any pair of vector fields $\boldsymbol{A}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$. In the present problem, you are asked to verify the above identity in the special case when the vector fields are the following plane-waves:

$$\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}_0 \exp[\mathrm{i}(\boldsymbol{k}_a\cdot\boldsymbol{r} - \omega_a t)], \qquad (1)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}_0 \exp[\mathrm{i}(\boldsymbol{k}_b\cdot\boldsymbol{r} - \omega_b t)]. \qquad (2)$$

In general, $\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{k}_a, \boldsymbol{k}_b$ are arbitrary complex vectors, while ω_a, ω_b are arbitrary complex numbers.

Hint: The vector identity $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ may be useful.

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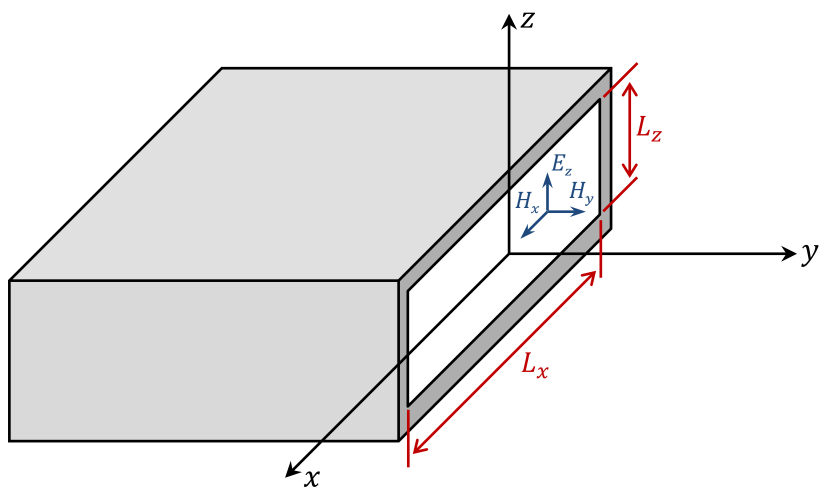
3 pts **Problem 2**) a) Show that the bound electric charge-density $\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$ and the bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t) = \partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t$, which are associated with polarization $\boldsymbol{P}(\boldsymbol{r},t)$, satisfy the charge-current continuity equation.

2 pts b) Show that $\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t) = 0$ and $\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)$, which are associated with magnetization $\boldsymbol{M}(\boldsymbol{r},t)$, satisfy the charge-current continuity equation.

Problem 3) An electromagnetic wave propagates along the y-axis in the free-space region inside the hollow rectangular cavity of a waveguide of width L_x and height L_z. The walls of the cavity are made of a perfectly electrically conducting metal. The guided wave has a single frequency ω and a k-vector $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$, with its E-field within the rectangular cavity specified as $\boldsymbol{E}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{z}}\cos(k_x x)\cos(k_y y - \omega t)$. Here E_0, k_x, k_y, and ω are real-valued constants.



1 pt a) Show that the E-field of the guided electromagnetic field satisfies Maxwell's first equation everywhere within the hollow cavity of the waveguide.

1 pt b) One of Maxwell's boundary conditions dictates that the tangential (or parallel) E-field at the inner surfaces of the waveguide must vanish. This condition imposes a constraint on the width L_x of the waveguide. Find the acceptable value(s) of L_x (as a function of k_x) that ensure the satisfaction of this boundary condition.

1 pt c) Another one of Maxwell's boundary conditions requires that the surface charge-density σ_s on the inner metallic surfaces be equal to $\varepsilon_0 E_\perp$, where $E_\perp$ is the perpendicular component of the E-field at the surface. Invoke this condition to determine the surface charge-density σ_s at the upper and lower surfaces of the cavity located at $z = \pm L_z/2$.

2 pts d) Use Maxwell's third equation, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, to determine the B-field everywhere inside the rectangular cavity of the waveguide.

1 pt e) Verify that the B-field obtained in part (d) satisfies Maxwell's fourth equation.

1 pt f) One of Maxwell's boundary conditions requires that $B_\perp$ at the inner surfaces of the metallic waveguide be zero. Show that the condition obtained in part (b) guarantees the satisfaction of this requirement.

2 pts g) Another one of Maxwell's boundary conditions relates the tangential component of the H-field at the inner metallic surfaces to the corresponding surface current-density $\boldsymbol{J}_s$. Invoke this condition to determine the surface current-density $\boldsymbol{J}_s$ at the upper and lower surfaces of the cavity located at $z = \pm L_z/2$.

2 pts h) Show that the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields inside the cavity satisfy Maxwell's second equation provided that $k_x^2 + k_y^2 = (\omega/c)^2$, where $c = 1/\sqrt{\mu_0 \varepsilon_0}$ is the speed of light in free space.

2 pts i) Verify that the surface charge and current densities obtained in parts (c) and (g) satisfy the charge-current continuity equation, $\boldsymbol{\nabla} \cdot \boldsymbol{J}_s + \partial \sigma_s/\partial t = 0$.

2 pts j) Find the Poynting vector $\boldsymbol{S}(\boldsymbol{r}, t)$ inside the cavity, then use it to explain (in words) how the electromagnetic energy moves forward, backward, and sideways at different points in spacetime.

Hint: In Cartesian coordinates, $\boldsymbol{\nabla} \times \boldsymbol{V} = (\partial V_z/\partial y - \partial V_y/\partial z)\hat{\boldsymbol{x}} + (\partial V_x/\partial z - \partial V_z/\partial x)\hat{\boldsymbol{y}} + (\partial V_y/\partial x - \partial V_x/\partial y)\hat{\boldsymbol{z}}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{V} = \partial V_x/\partial x + \partial V_y/\partial y + \partial V_z/\partial z$.
