Opti 501 **1^{st} Midterm Exam** (10/3/2019) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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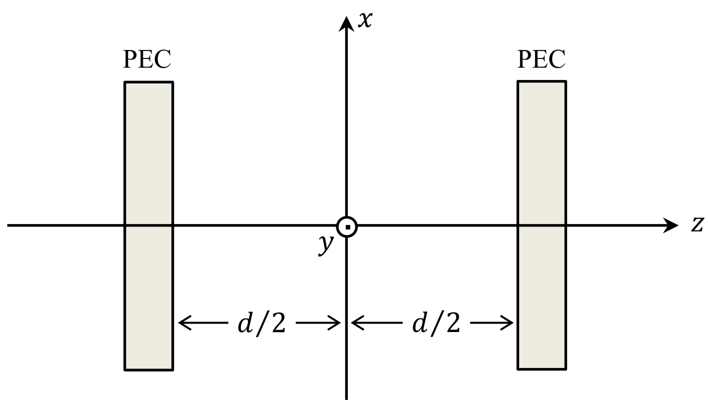
5 pts **Problem 1**) The scalar field $\psi(\boldsymbol{r},t)$ and the vector field $\boldsymbol{A}(\boldsymbol{r},t)$ are specified in the spherical coordinate system $\boldsymbol{r}=(r,\theta,\varphi)$. Using the divergence and gradient operators in spherical coordinates, prove the vector identity $\boldsymbol{\nabla}\cdot(\psi\boldsymbol{A})=\psi\boldsymbol{\nabla}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{\nabla}\psi$.

Hint: In spherical coordinates, the divergence and gradient operators are given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=\frac{\partial(r^2V_r)}{r^2\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta V_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi},$$

$$\boldsymbol{\nabla}f=\frac{\partial f}{\partial r}\hat{\boldsymbol{r}}+\frac{\partial f}{r\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\boldsymbol{\varphi}}.$$

Problem 2) A pair of infinitely large, perfectly electrically conducting (PEC) mirrors is placed parallel to the xy-plane at $z=\pm d/2$, as shown. In the free-space region $|z|\le d/2$, the E-field distribution is $\boldsymbol{E}(\boldsymbol{r},t)=E_0\hat{\boldsymbol{x}}\sin(\omega z/c)\sin(\omega t)$, while the corresponding H-field distribution is $\boldsymbol{H}(\boldsymbol{r},t)=(E_0/Z_0)\hat{\boldsymbol{y}}\cos(\omega z/c)\cos(\omega t)$. The vacuum wavelength of the electromagnetic field trapped between the mirrors is $\lambda_0=2\pi c/\omega$, and the gap between the mirrors is an integer-multiple of the wavelength, that is, $d=n\lambda_0$, where n is an arbitrary positive integer.



6 pts a) Confirm that, in the region between the mirrors, all four equations of Maxwell are satisfied.

3 pts b) Show that the E-field at the interior surfaces of both mirrors (i.e., at $z=\pm d/2$) satisfies Maxwell's boundary conditions.

3 pts c) Considering that the H-field is also required to satisfy Maxwell's boundary conditions, find the surface current-density $\boldsymbol{J}_s$ at the interior facets of both mirrors (i.e., at $z=\pm d/2$).

Hint: In Cartesian coordinates,

$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z},$$

$$\boldsymbol{\nabla}\times\boldsymbol{V}=\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\hat{\boldsymbol{z}}.$$

Problem 3) Maxwell's first equation relates the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ to the free charge-density $\rho_{\text{free}}(\boldsymbol{r},t)$ and the bound charge-density $\rho_{\text{bound}}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$. Let a complex-valued electric field $\boldsymbol{E} = \boldsymbol{E}' + \mathrm{i}\boldsymbol{E}''$ satisfy Maxwell's first equation in conjunction with the complex-valued sources $\rho_{\text{free}} = \rho' + \mathrm{i}\rho''$ and $\boldsymbol{P} = \boldsymbol{P}' + \mathrm{i}\boldsymbol{P}''$, where the E-field as well as the source distributions are complex functions of the spacetime $(\boldsymbol{r},t)$.

2 pts a) What E-field distribution would satisfy Maxwell's 1st equation if the sources were replaced by the complex conjugates $\rho^*_{\text{free}} = \rho' - \mathrm{i}\rho''$ and $\boldsymbol{P}^* = \boldsymbol{P}' - \mathrm{i}\boldsymbol{P}''$ of the aforementioned sources?

2 pts b) What E-field distribution would satisfy Maxwell's 1st equation if the sources were replaced by $\mathrm{Real}(\rho_{\text{free}}) = \rho'$ and $\mathrm{Real}(\boldsymbol{P}) = \boldsymbol{P}'$?

2 pts c) What E-field distribution would satisfy Maxwell's 1st equation if the sources were replaced by $\mathrm{Imag}(\rho_{\text{free}}) = \rho''$ and $\mathrm{Imag}(\boldsymbol{P}) = \boldsymbol{P}''$?

2 pts d) Generalize the statement of the problem and answer the above questions once again in the case that the specified sources are ρ_{free}, $\boldsymbol{J}_{\text{free}}$, $\boldsymbol{P}$, and $\boldsymbol{M}$, the electromagnetic fields of interest are $\boldsymbol{E}$ and $\boldsymbol{H}$, and the equations that must be simultaneously satisfied are all four of Maxwell's equations.
