Opti 501 **1^{st} Midterm Exam** (9/28/2017) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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6 pts **Problem 1**) The E-field of a plane-wave propagating in free space is written as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = \exp(\gamma t - \beta_1 x - \beta_2 y - \beta_3 z)$$
$$\times [\boldsymbol{A}\cos(\alpha_1 x + \alpha_2 y + \alpha_3 z - \omega_0 t + \varphi_A) - \boldsymbol{B}\sin(\alpha_1 x + \alpha_2 y + \alpha_3 z - \omega_0 t + \varphi_B)].$$

In the above expression, $\boldsymbol{\alpha} = \alpha_1\hat{\boldsymbol{x}} + \alpha_2\hat{\boldsymbol{y}} + \alpha_3\hat{\boldsymbol{z}}$, $\boldsymbol{\beta} = \beta_1\hat{\boldsymbol{x}} + \beta_2\hat{\boldsymbol{y}} + \beta_3\hat{\boldsymbol{z}}$, $\boldsymbol{A} = A_1\hat{\boldsymbol{x}} + A_2\hat{\boldsymbol{y}} + A_3\hat{\boldsymbol{z}}$, and $\boldsymbol{B} = B_1\hat{\boldsymbol{x}} + B_2\hat{\boldsymbol{y}} + B_3\hat{\boldsymbol{z}}$ are constant real-valued vectors, whereas the parameters γ, ω_0, φ_A, and φ_B are constant real-valued scalars. By comparing the above expression with the complex representation of the same E-field, namely, $\boldsymbol{E}(\boldsymbol{r},t) = \text{Real}\{\boldsymbol{E}_0 \exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}$, identify the complex-valued $\boldsymbol{E}_0$, $\boldsymbol{k}$, and ω.

Problem 2) The static current-density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r})$ resides in free space; all other sources of the electromagnetic field are absent, that is, $\rho_{\text{free}}(\boldsymbol{r},t) = \boldsymbol{P}(\boldsymbol{r},t) = \boldsymbol{M}(\boldsymbol{r},t) = 0$. In the (ρ,φ,z) cylindrical coordinate system, the resulting B-field is written

$$\boldsymbol{B}(\boldsymbol{r}) = B_0\{(\rho z/z_0^2)\hat{\boldsymbol{\rho}} - [(\rho/\rho_0)^2 - 1]\hat{\boldsymbol{z}}\}\exp[-(\rho/\rho_0)^2 - (z/z_0)^2],$$

where B_0, ρ_0, and z_0 are arbitrary constants.

2 pts a) Show that the above B-field satisfies Maxwell's 4^{th} equation, that is, $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r}) = 0$.

3 pts b) Use Maxwell's 2^{nd} equation, $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r})$, to determine the free current-density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r})$ that has given rise to the specified $\boldsymbol{B}(\boldsymbol{r})$.

1 pt c) Show that the charge-current continuity equation is satisfied everywhere, that is, in the absence of $\rho_{\text{free}}(\boldsymbol{r},t)$, confirm that $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r}) = 0$.

Problem 3) The static charge-density $\rho_{\text{free}}(\boldsymbol{r})$ resides in free space; all other sources of the electromagnetic field are absent, that is, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = \boldsymbol{P}(\boldsymbol{r},t) = \boldsymbol{M}(\boldsymbol{r},t) = 0$. In the (r,θ,φ) spherical coordinate system, the resulting E-field is written

$$\boldsymbol{E}(\boldsymbol{r}) = E_0\left[2(r/r_0)\sin\theta\,\hat{\boldsymbol{r}} - (r_0/r)\cos\theta\,\hat{\boldsymbol{\theta}}\right]\exp[-(r/r_0)^2],$$

where E_0 and r_0 are arbitrary constants.

2 pts a) Show that the above E-field satisfies Maxwell's 3^{rd} equation, that is, $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r}) = 0$.

2 pts b) Use Maxwell's 1^{st} equation, $\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r}) = \rho_{\text{free}}(\boldsymbol{r})$, to determine the free charge-density $\rho_{\text{free}}(\boldsymbol{r})$ that has given rise to the specified $\boldsymbol{E}(\boldsymbol{r})$.

2 pts c) Find the total charge by integrating $\rho_{\text{free}}(\boldsymbol{r})$ over the entire space.

Hint: $\int_0^\infty x^2\exp(-x^2)\,\text{d}x = \sqrt{\pi}/4$; $\int_0^\infty x^4\exp(-x^2)\,\text{d}x = 3\sqrt{\pi}/8$. (G&R 3.461-2)

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1 pt **Problem 4**) a) Write Maxwell's equations in their standard differential form. Also write the standard expressions for the displacement field $\boldsymbol{D}(\boldsymbol{r},t)$ and the magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$ in terms of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as well as the polarization $\boldsymbol{P}(\boldsymbol{r},t)$ and magnetization $\boldsymbol{M}(\boldsymbol{r},t)$.

2 pts b) Eliminate the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields from Maxwell's equations, leaving only the $\boldsymbol{D}$ and $\boldsymbol{B}$ fields.

2 pts c) Explain the type of *total* charge-density and *total* current-density that appear in Maxwell's equations when they are written as in part (b). Pay particular attention to the units (i.e., dimensions) of the various entities.

2 pts d) Derive an alternative version of Poynting's theorem in which the electromagnetic energy flux is *defined* as $\boldsymbol{S}(\boldsymbol{r},t) = c^2\boldsymbol{D}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t)$. In accordance with this alternative form of Poynting's theorem, identify the corresponding field energy-densities and the exchange rate of electromagnetic energy with material media.

Appendix B

Vector Identities

$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A}$

$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$

$\boldsymbol{A}\times\boldsymbol{B}=-\boldsymbol{B}\times\boldsymbol{A}$

$\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C}$

$(\boldsymbol{A}+\boldsymbol{B})\times\boldsymbol{C}=\boldsymbol{A}\times\boldsymbol{C}+\boldsymbol{B}\times\boldsymbol{C}$

$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$

$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$

$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$

$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})=0$

$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi)=0$

$\boldsymbol{\nabla}\cdot(\psi\boldsymbol{A})=(\boldsymbol{\nabla}\psi)\cdot\boldsymbol{A}+\psi\boldsymbol{\nabla}\cdot\boldsymbol{A}$

$\boldsymbol{\nabla}\times(\psi\boldsymbol{A})=(\boldsymbol{\nabla}\psi)\times\boldsymbol{A}+\psi\boldsymbol{\nabla}\times\boldsymbol{A}$

$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})-\boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$

$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\boldsymbol{\nabla}^2\boldsymbol{A}$

$\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B})=(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}+\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})+\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A})$

$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B})-\boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A})+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}-(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}$

If $\boldsymbol{r}$ is a point in 3-dimensional Euclidean space, then

$\boldsymbol{\nabla}(1/|\boldsymbol{r}|)=-\boldsymbol{r}/|\boldsymbol{r}|^3$

$\boldsymbol{\nabla}\cdot(\boldsymbol{r}/|\boldsymbol{r}|^3)=4\pi\delta(\boldsymbol{r})$

Appendix C

Vector Operations in Cartesian, Cylindrical, and Spherical Coordinates

Cartesian (x, y, z):

$$\boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\psi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla}^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

Cylindrical (ρ, ϕ, z):

$$\boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{\partial\psi}{\rho\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial(\rho A_\rho)}{\rho\partial\rho} + \frac{\partial A_\phi}{\rho\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \left(\frac{\partial A_z}{\rho\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla}^2\psi = \frac{\partial(\rho\,\partial\psi/\partial\rho)}{\rho\partial\rho} + \frac{\partial^2\psi}{\rho^2\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ):

$$\boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial(r^2 A_r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{r\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$$

$$\boldsymbol{\nabla}^2\psi = \frac{\partial(r^2\partial\psi/\partial r)}{r^2\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial(\sin\theta\,\partial\psi/\partial\theta)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{\partial(r^2\partial\psi/\partial r)}{r^2\partial r} = \frac{\partial^2(r\psi)}{r\partial r^2}\right]$$

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