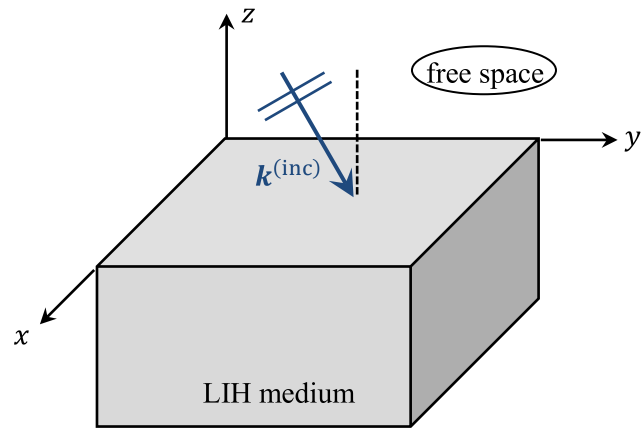
**Opti 501** **Final Exam** (12/14/2021) **Time: 2 hours**

**Please write your name and ID number on all the pages, then staple them together.**
**Answer all the questions.**

**Note: Bold symbols represent vectors and vector fields.**

**Problem 1**) Suppose a homogeneous plane-wave (i.e., one whose $k$-vector is real) arrives from the free space onto the flat and polished surface of a linear, isotropic, and homogeneous (LIH) medium, as shown. Let the interface between the LIH medium and the medium of incidence (i.e., free space in the present case) be the $xy$-plane at $z = 0$. Using brief but precise statements, define the following properties of the optical material, characteristics of the plane-wave, and specific features of the optical system.




3 pts a) When is an optical medium considered to be linear, isotropic, and homogeneous (LIH)?

2 pts b) What is the plane of incidence? Does this definition hold for a normally-incident plane-wave?

2 pts c) When is the incident plane-wave said to be $p$-polarized? When is it said to be $s$-polarized?

2 pts d) Denoting the components of the incident $E$-field by $E_p = |E_p|e^{i\varphi_p}$ and $E_s = |E_s|e^{i\varphi_s}$, describe conditions under which the incident plane-wave can be said to be linearly polarized, or circularly polarized, or elliptically polarized.

**Problem 2**) Inside a linear, isotropic, and homogenous medium, the complex $E$-field of a plane electromagnetic wave is generally written as $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$, where $\boldsymbol{E}_0 = \boldsymbol{E}_0' + i\boldsymbol{E}_0''$ and $\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}''$ are constant complex vectors, $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ is a real vector that identifies the observation point within a Cartesian coordinate system, and $t$ is a real variable that specifies the observation time. In general, $\omega$ is a complex parameter representing the oscillation frequency of the plane-wave, although, in the present problem, you may assume that $\omega$ is real-valued.

2 pts a) Considering that the actual $E$-field is the real part of the above $\boldsymbol{E}(\boldsymbol{r}, t)$, write an expression for the actual $E$-field in terms of $\boldsymbol{E}_0', \boldsymbol{E}_0'', \boldsymbol{k}', \boldsymbol{k}'', \omega, \boldsymbol{r}$, and $t$.

2 pts b) In the expression obtained in part (a), what are the direction and the rate of attenuation of the $E$-field in the three-dimensional space specified by the position vector $\boldsymbol{r}$?

2 pts c) In the expression obtained in part (a), what are the direction and the speed of propagation of the $E$-field's phase-fronts in the three-dimensional space specified by the position vector $\boldsymbol{r}$?

2 pts d) In accordance with the expression obtained in part (a), what determines the state of polarization of the plane-wave? When is the plane-wave linearly-polarized? Under what circumstances can one say that the plane-wave is circularly-polarized?

2 pts e) Under what circumstances can one say that the plane-wave is homogeneous? Explain the nature of the physical differences between homogeneous and inhomogeneous plane-waves.

2 pts **Problem 3**) a) For an electromagnetic plane-wave residing inside a linear, isotropic, and homogeneous medium, write a (generally complex) pair of expressions for the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$ in terms of $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{k}$, and $\omega$.

3 pts b) Assuming that $\rho_{\text{free}}(\boldsymbol{r},t)=0$ and $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)=0$, write all four equations of Maxwell for the $E$-field and $H$-field of the plane-wave, keeping in mind that the remaining sources within the medium should be expressed as $\boldsymbol{P}(\boldsymbol{r},t)=\varepsilon_0\chi_{\text{e}}(\omega)\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{M}(\boldsymbol{r},t)=\mu_0\chi_{\text{m}}(\omega)\boldsymbol{H}(\boldsymbol{r},t)$. Simplify these equations by eliminating the $\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$, $\partial/\partial t$ operators and using the material medium's (relative) permittivity $\varepsilon(\omega)=1+\chi_{\text{e}}(\omega)$ and permeability $\mu(\omega)=1+\chi_{\text{m}}(\omega)$.

2 pts c) Use Maxwell's 1st equation obtained in part (b) to express $E_{0z}$ in terms of $E_{0x}, E_{0y}, k_x, k_y, k_z$.

3 pts d) Derive the dispersion relation $\boldsymbol{k}\cdot\boldsymbol{k}=(\omega/c)^2\,\mu(\omega)\varepsilon(\omega)$ from the Maxwell equations obtained in part (b). How is the material medium's (complex) refractive index $n(\omega)$ related to its (relative) permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$?

**Hint**: The vector identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ should be helpful.

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**Problem 4**) The Poynting vector is defined as $\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)$, where $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{S}$ at the arbitrary point $(\boldsymbol{r},t)$ in spacetime are intended to be real-valued. When complex notation is used for the electromagnetic fields, one can write $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}'(\boldsymbol{r},t)+\mathrm{i}\boldsymbol{E}''(\boldsymbol{r},t)$, where $\boldsymbol{E}'$ and $\boldsymbol{E}''$ are the real and imaginary parts of the complex $E$-field. Similarly, one writes $\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}'(\boldsymbol{r},t)+\mathrm{i}\boldsymbol{H}''(\boldsymbol{r},t)$, with $\boldsymbol{H}'$ and $\boldsymbol{H}''$ being the real and imaginary parts of the $H$-field.

2 pts a) What is the correct expression for the Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$ in terms of $\boldsymbol{E}', \boldsymbol{E}'', \boldsymbol{H}'$, and $\boldsymbol{H}''$?

2 pts b) When complex notation is used for $\boldsymbol{E}$ and $\boldsymbol{H}$ fields, the *incorrect* way to express the Poynting vector is $\boldsymbol{S}(\boldsymbol{r},t)=\text{Real}\{\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\}$. Write this incorrect expression for $\boldsymbol{S}(\boldsymbol{r},t)$ in terms of $\boldsymbol{E}', \boldsymbol{E}'', \boldsymbol{H}', \boldsymbol{H}''$, then confirm that indeed it differs from the correct expression found in (a).

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**Problem 5**) A linear, isotropic, homogeneous material medium is said to be transparent when its refractive index $n(\omega)$ is real-valued. Let a plane-wave described by its electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$ and $\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$ reside within this transparent medium. In general, $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{k}$, and $\omega$ are complex entities, but, for purposes of this problem, you may assume that $\omega$ is real-valued. When the plane-wave is homogeneous (i.e., when $\boldsymbol{k}''=0$), we say the plane-wave is of "propagating" type, and when it is inhomogeneous (i.e., when $\boldsymbol{k}''\neq 0$) while the host medium is transparent, we say the plane-wave is "evanescent."

2 pts a) Use the dispersion relation to show that, for evanescent waves, $\boldsymbol{k}'$ and $\boldsymbol{k}''$ must be orthogonal to each other (that is, $\boldsymbol{k}'\cdot\boldsymbol{k}''=0$) and, moreover, $|\boldsymbol{k}'|>|\boldsymbol{k}''|$.

2 pts b) Invoke Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$, to express $\boldsymbol{H}_0$ in terms of $\boldsymbol{k}, \boldsymbol{E}_0$, $\omega, \mu_0$, and $\mu(\omega)$. [Note that, for transparent materials, $\mu(\omega)$ and $\varepsilon(\omega)$ must be real-valued.]

3 pts c) Evaluate the time-averaged Poynting vector $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=½\text{Real}\{\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}^*(\boldsymbol{r},t)\}$, then show that the component of this time-averaged Poynting vector along the direction of $\boldsymbol{k}''$ vanishes.

**Hint**: The vector identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ should be helpful.

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