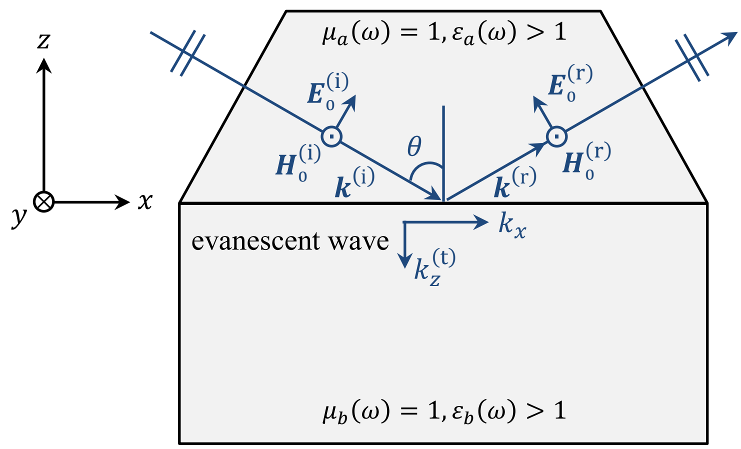
**Opti 501** **Final Exam** (12/15/2020) **Time: 2 hours**

**Please write your name and ID number on the first page before scanning/photographing the pages. Answer all the questions.**

**Note: Bold symbols represent vectors and vector fields.**

**Problem 1**) A homogeneous plane-wave of frequency $\omega$ arrives at the interface between two linear, isotropic, homogeneous media at an incidence angle $\theta$ that is greater than the critical angle $\theta_c$ for total internal reflection. The plane of incidence is $xz$, the incident beam is $p$-polarized, and the incidence and transmittance media have permeability $\mu_a(\omega) = 1$, $\mu_b(\omega) = 1$, and real-valued permittivity $\varepsilon_a(\omega) > \varepsilon_b(\omega) > 1$. The magnetic field within the transmittance medium, being oriented along the $y$-axis, is written as $\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r}, t) = H_{oy}\hat{\boldsymbol{y}} \exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{t})} \cdot \boldsymbol{r} - \omega t)]$.




4 pts a) Write the expressions of $\boldsymbol{k}^{(\mathrm{i})}$, $\boldsymbol{k}^{(\mathrm{r})}$, and $\boldsymbol{k}^{(\mathrm{t})}$ for the incident, reflected, and transmitted $k$-vectors in terms of $\omega$, the speed of light in vacuum, $c$, and the refractive indices $n_a(\omega)$ and $n_b(\omega)$ of the two media.

4 pts b) Use the dispersion relation in the transmittance medium to derive $k_z^{(\mathrm{t})}$ as a function of the incidence angle $\theta$, the frequency $\omega$, the speed of light in vacuum, $c$, and the refractive indices $n_a(\omega)$ and $n_b(\omega)$ of the two media. Explain why $k_z^{(\mathrm{t})}$ should be a negative imaginary number.

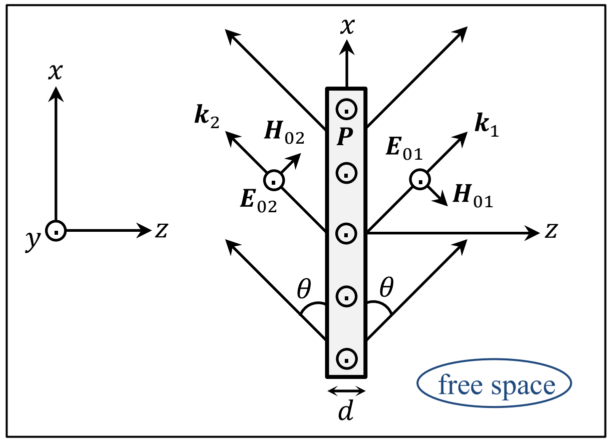
2 pts c) Confirm that the transmitted wave satisfies Maxwell's 4$^{\text{th}}$ equation, namely, $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0$.

4 pts d) In the absence of free currents (i.e., $\boldsymbol{J}_{\text{free}} = 0$), use Maxwell's 2$^{\text{nd}}$ equation, $\boldsymbol{\nabla} \times \boldsymbol{H} = \partial_t \boldsymbol{D}$, to express the electric field $\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r}, t)$ of the evanescent wave within the transmittance medium in terms of the corresponding magnetic field amplitude $H_{oy}^{(\mathrm{t})}$ and the various system parameters.

3 pts e) Confirm that the transmitted wave satisfies Maxwell's 1$^{\text{st}}$ and 3$^{\text{rd}}$ equations as well.

3 pts f) Show that the time-averaged Poynting vector $\langle \boldsymbol{S}(\boldsymbol{r}, t) \rangle$ of the evanescent wave has a nonzero component along the $x$-axis, while its components along the $y$ and $z$ axes are precisely zero. Express the $x$-component of the time-averaged Poynting vector in terms $H_{oy}^{(\mathrm{t})}$ and the various system parameters.

**Problem 2**) A large, thin sheet of dielectric material has (very small) thickness $d$, relative permittivity $\varepsilon(\omega)$, and relative permeability $\mu(\omega) = 1$. The sheet sits in the $xy$-plane of a Cartesian coordinate system at $z = 0$, where its electric dipoles oscillate at a fixed frequency $\omega$ along the $y$-axis. The material polarization is




$$\boldsymbol{P}(\boldsymbol{r}, t) = P_0 \hat{\boldsymbol{y}} \cos(\kappa_0 x - \omega t - \varphi_0),$$

where $P_0$, $\kappa_0$, $\omega$ and $\varphi_0$ are positive real-valued constants. The oscillating dipoles radiate a pair of $s$-polarized plane-waves into the right half-space $z > 0$ and left half-space $z < 0$, as shown. Aside from their propagation directions, which are at $\pm\theta$ relative to the $x$-axis within the $xz$-plane, the two radiated plane-waves are identical in every respect.

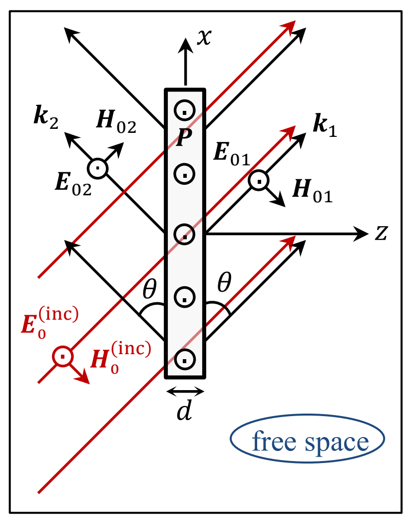
4 pts a) In terms of $\omega$, $\theta$, $E_0 = |E_{01}| = |E_{02}|$, $c = (\mu_0\varepsilon_0)^{-1/2}$, and $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$, write expressions for $\boldsymbol{k}_1$, $\boldsymbol{E}_1(\boldsymbol{r}, t)$, $\boldsymbol{H}_1(\boldsymbol{r}, t)$ and $\boldsymbol{k}_2$, $\boldsymbol{E}_2(\boldsymbol{r}, t)$, $\boldsymbol{H}_2(\boldsymbol{r}, t)$.

4 pts b) Invoking Maxwell's 2nd equation, $\boldsymbol{\nabla} \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$, along with the fact that the thickness $d$ of the sheet is very small, show that the spatial frequency $\kappa_0$ of the polarization $\boldsymbol{P}(\boldsymbol{r}, t)$ along the $x$-axis is given by $\kappa_0 = (\omega/c)\cos\theta$.

**Warning**: Do *not* attempt to relate the $D$-field inside the material medium to the local $E$-field via the permittivity $\varepsilon_0\varepsilon(\omega)$. This is because the dipoles are being driven by metaphorical "ants" at this point. Later, when an incident plane-wave is brought in to drive the dipoles, you will be able to invoke the constitutive relation $\boldsymbol{D} = \varepsilon_0\varepsilon(\omega)\boldsymbol{E}$.

4 pts c) Confirm that $\boldsymbol{E}_{\parallel}$ and $\boldsymbol{B}_{\perp}$ are continuous across the dielectric sheet, then relate the discontinuity of $\boldsymbol{H}_{\parallel}$ to the time-derivative of the $D$-field inside the material.

4 pts d) Let the electric dipoles be driven by the incident $E$-field $\boldsymbol{E}_0^{(\text{inc})} \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$ of an $s$-polarized plane-wave whose $k$-vector coincides with $\boldsymbol{k}_1$. Inside the dielectric sheet, we have




$$\boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon_0\varepsilon(\omega)(E_0^{(\text{inc})} + E_0)\hat{\boldsymbol{y}} \exp[\mathrm{i}(\kappa_0 x - \omega t)].$$

Invoking $\boldsymbol{\nabla} \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$ to relate the discontinuity of $\boldsymbol{H}_{\parallel}$ across the sheet to the time-derivative of the above $D$-field, find the reflection coefficient $E_0/E_0^{(\text{inc})}$.

4 pts e) Considering that the transmitted $E$-field can be written as $(\boldsymbol{E}_0^{(\text{inc})} + \boldsymbol{E}_{01}) \exp[\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{r} - \omega t)]$, find the transmission coefficient for the $E$-field.

**Hint**: $\boldsymbol{\nabla} \times \boldsymbol{H} = (\partial_y H_z - \partial_z H_y)\hat{\boldsymbol{x}} + (\partial_z H_x - \partial_x H_z)\hat{\boldsymbol{y}} + (\partial_x H_y - \partial_y H_x)\hat{\boldsymbol{z}}$.
For the partial derivative $\partial_z H_x$, since $H_x$ is discontinuous at $z = 0$, use $\partial_z H_x \cong \Delta H_x/\Delta z = \Delta H_x/d$.