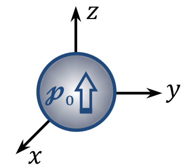
Opti 501 **Final Exam** (12/17/2019) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) A solid spherical particle of radius R, volume $v = 4\pi R^3/3$, and uniform polarization $P_0\hat{\boldsymbol{z}}$ (corresponding to an electric dipole moment $\boldsymbol{p}_0 = P_0 v\hat{\boldsymbol{z}}$), is centered at the origin of coordinates in free space, producing an electric field inside and outside the sphere, as follows:

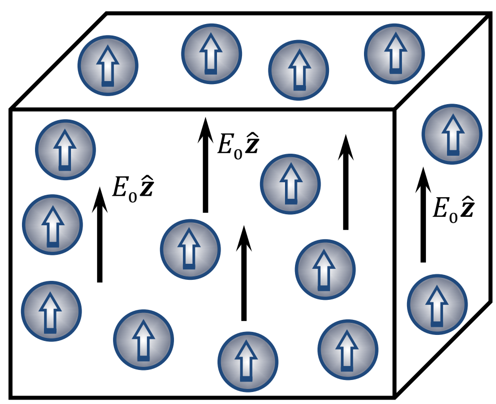
$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} -\dfrac{P_0\hat{\boldsymbol{z}}}{3\varepsilon_0}; & r < R, \\ \dfrac{p_0}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right); & r > R. \end{cases}$$



4 pts a) Show that the integral of the E-field *outside* the sphere (i.e., in the region $r > R$) equals zero. (This result will be useful later on, when you'll argue that the *average* E-field produced by a large number of randomly distributed spherical dipoles *outside* the spheres can be set to zero.)

2 pts b) Let the particle be linearly polarizable with a polarizability coefficient ζ, so that, in the presence of a constant, uniform, externally-applied electric field $E_0\hat{\boldsymbol{z}}$, the particle's electric dipole moment is given by $\boldsymbol{p}_0 = \zeta E_0\hat{\boldsymbol{z}}$. What are the units (or dimensions) of ζ?

4 pts c) A large number of identical spherical particles are randomly distributed throughout free space, and a constant, uniform electric field $E_0\hat{\boldsymbol{z}}$ is applied. The particles have radius R, polarizability coefficient ζ, and number-density N (i.e., number of particles per unit volume is N). What volume fraction of space do the particles occupy? What is the *average* E-field within a unit volume of space?



4 pts d) Considering that the spatially-averaged polarization over a unit volume of space is $\langle \boldsymbol{P}(\boldsymbol{r})\rangle = N\zeta E_0\hat{\boldsymbol{z}}$, and that you have already found an expression for the average E-field (over a unit volume) in part (c), what is the dielectric susceptibility χ_e of this "gas" of identical (spherical) particles?

3 pts e) Generalize the result obtained in part (d) to the case where the gas is a mixture of spherical particles of radii $R_1, R_2, \cdots, R_K$, polarizability coefficients $\zeta_1, \zeta_2, \cdots, \zeta_K$, and number-densities $N_1, N_2, \cdots, N_K$.

3 pts f) Compare the results of this problem with the Clausius-Mossotti correction to the Lorentz oscillator model by pointing out the similarities and differences between the two.

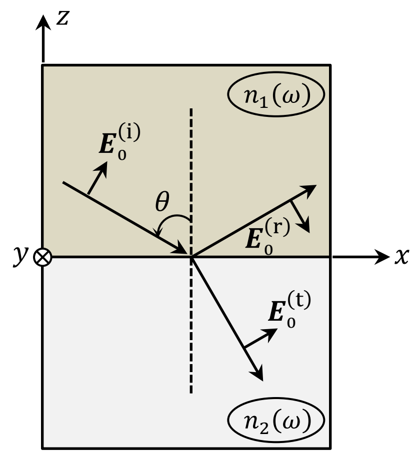
Hint: In spherical coordinates,

$$\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\,\hat{\boldsymbol{x}} + \sin\theta\sin\varphi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\boldsymbol{x}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}},$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\boldsymbol{x}} + \cos\varphi\,\hat{\boldsymbol{y}}.$$

Problem 2) A *p*-polarized plane wave, having frequency ω, arrives at oblique incidence at the interface between a transparent medium of refractive index $n_1(\omega)$ and a second transparent medium of refractive index $n_2(\omega)$. It is assumed here that $\mu_1(\omega) = \mu_2(\omega) = 1$, and that both $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$ are real-valued and greater than 1. In the xz plane of incidence, the incidence angle is θ, and the incident E-field is $E_{0x}^{(\mathrm{i})}\hat{\boldsymbol{x}} + E_{0z}^{(\mathrm{i})}\hat{\boldsymbol{z}}$. To simplify the notation, you may use $c = 1/\sqrt{\mu_0\varepsilon_0}$ for the speed of light in vacuum, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ for the impedance of free space, and $k_0 = \omega/c$ for the vacuum wave-number associated with each plane-wave.



2 pts a) Invoking Snell's law and the dispersion relation $\boldsymbol{k}\cdot\boldsymbol{k} = k^2 = k_0^2 n^2(\omega)$, identify the k-vectors $\boldsymbol{k}^{(\mathrm{i})}$, $\boldsymbol{k}^{(\mathrm{r})}$, and $\boldsymbol{k}^{(\mathrm{t})}$ of the three plane-waves.

2 pts b) Use Maxwell's first equation to relate E_{0z} of each plane-wave to the corresponding E_{0x}, the incidence angle θ, and the refractive indices n_1 and n_2.

3 pts c) Use Maxwell's third equation to express $\boldsymbol{H}_0^{(\mathrm{i})}$, $\boldsymbol{H}_0^{(\mathrm{r})}$, and $\boldsymbol{H}_0^{(\mathrm{t})}$ in terms of the corresponding E_{0x}, the incidence angle θ, the free-space impedance Z_0, and the refractive indices n_1 and n_2.

3 pts d) Match the boundary conditions at the interfacial plane at $z = 0$ to arrive at Fresnel's formulas for the reflection coefficient $\rho_p = E_{0x}^{(\mathrm{r})}/E_{0x}^{(\mathrm{i})}$ and transmission coefficient $\tau_p = E_{0x}^{(\mathrm{t})}/E_{0x}^{(\mathrm{i})}$.

2 pts e) Find the formula for the Brewster angle θ_B, which is the incidence angle at which $\rho_p = 0$.

3 pts f) Under what circumstances does total internal reflection (TIR) occur? In what ways do $k_z^{(\mathrm{t})}$, $E_{0z}^{(\mathrm{t})}$, and $\boldsymbol{H}_0^{(\mathrm{t})}$ change when θ goes from below to above the critical TIR angle θ_c?

3 pts g) Use the results obtained in parts (a)–(d) to relate the Poynting vector $\boldsymbol{S}^{(\mathrm{t})}(x, z, t)$ of the transmitted plane-wave to θ, n_1, n_2, ω, k_0, and $E_{0x}^{(\mathrm{t})}$. (**Hint**: Cases where TIR does and does not occur require separate treatments.)

2 pts h) In the case of total internal reflection, confirm that the time-averaged z-component of the transmitted Poynting vector would vanish, that is, $\langle S_z^{(\mathrm{t})}\rangle = 0$.