Opti 501 **Final Exam** (12/11/2018) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) A monochromatic plane-wave of frequency ω and wave-vector $\boldsymbol{k} = (\omega/c)n(\omega)\hat{\boldsymbol{z}}$ propagates along the z-axis within a linear, homogeneous, isotropic medium of refractive index $n(\omega)$. The E-field amplitude of the plane-wave is given by the complex vector $\boldsymbol{E}_0 = \boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0''$.

3 pts a) Show that both $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are confined to the xy-plane; in other words, $E_{zo}' = E_{zo}'' = 0$.

3 pts b) In terms of the remaining components $E_{xo}', E_{xo}'', E_{yo}', E_{yo}''$ of $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$, write expressions for the *magnitude* and *phase* of the E-field amplitudes along the x and y axes, namely, $|E_{xo}|\exp(\mathrm{i}\varphi_{xo})$ and $|E_{yo}|\exp(\mathrm{i}\varphi_{yo})$.

4 pts c) The plane-wave is said to be linearly polarized when $\varphi_{xo} - \varphi_{yo}$ equals zero or $\pm 180°$. Under such circumstances, show that $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ must be parallel (or anti-parallel) to each other.

4 pts d) The plane-wave is said to be circularly polarized when $|E_{xo}| = |E_{yo}|$ and $\varphi_{xo} - \varphi_{yo} = \pm 90°$. Under these circumstances, show that the vectors $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ have equal lengths and are perpendicular to each other.

Hint: $\tan(\varphi_{xo} - \varphi_{yo}) = [\tan(\varphi_{xo}) - \tan(\varphi_{yo})]/[1 + \tan(\varphi_{xo})\tan(\varphi_{yo})]$.

Problem 2) In a linear, homogeneous, isotropic medium, a monochromatic electromagnetic field of frequency ω has its displacement given by $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\varepsilon(\omega)\boldsymbol{E}(\boldsymbol{r},t)$ and its magnetic induction given by $\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\mu(\omega)\boldsymbol{H}(\boldsymbol{r},t)$. Defining the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ via $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B}$, and the scalar potential $\psi(\boldsymbol{r},t)$ via $\boldsymbol{E} = -\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t$, ensures the satisfaction of Maxwell's third and fourth equations.

6 pts a) Assuming that $\rho_{\text{free}}(\boldsymbol{r},t) = 0$ and $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = 0$, and working in an appropriate version of the Lorenz gauge, use Maxwell's first and second equations to arrive at two wave equations (i.e., second-order partial differential equations), one for $\boldsymbol{A}(\boldsymbol{r},t)$ and another for $\psi(\boldsymbol{r},t)$.

4 pts b) For plane-waves $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ and $\psi(\boldsymbol{r},t) = \psi_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, use the wave equations obtained in (a) to arrive at the dispersion relation, which relates the wave-vector $\boldsymbol{k}$ to ω, $n(\omega) = \sqrt{\mu(\omega)\varepsilon(\omega)}$, and the speed of light in vacuum, $c = 1/\sqrt{\mu_0\varepsilon_0}$.

4 pts c) Relate the E-field and B-field amplitudes $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ of the corresponding plane-waves to the amplitudes ψ_0 and $\boldsymbol{A}_0$ of the scalar and vector potentials described in (b).

Hint: $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}$ and $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi) = \nabla^2\psi$.

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12 pts **Problem 3**) Two identical, flat, partially-transmissive mirrors are placed back to back in free space, with a gap d separating the two, as shown in the figure. At normal incidence, each mirror has Fresnel reflection coefficient ρ_m and transmission coefficient τ_m. Also, each mirror is constructed such that, whether the incident beam arrives from the left-hand side or from the right-hand side, its reflection coefficients from the front-side and the back-side are identical, as are its corresponding front-side and back-side transmission coefficients.

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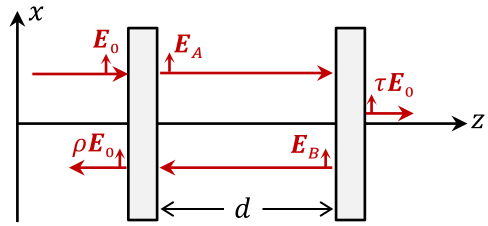

The figure shows a monochromatic, linearly-polarized plane-wave of frequency ω and E-field amplitude $\boldsymbol{E}_0 = E_0\hat{\boldsymbol{x}}$ arriving at the front facet of the first mirror at normal incidence, creating a pair of counter-propagating plane-waves with amplitudes $\boldsymbol{E}_A$ and $\boldsymbol{E}_B$ inside the gap, then emerging with amplitude $\tau\boldsymbol{E}_0$ from the rear facet of the second mirror. The total E-field amplitude returning from the front facet of the first mirror is denoted by $\rho\boldsymbol{E}_0$. Find the overall reflection and transmission coefficients (i.e., ρ and τ) of the Fabry-Perot cavity thus formed by the pair of parallel (partially-transmissive) mirrors.

Hint: The various parameters shown in the figure are interrelated. For instance, $\tau E_0 = \tau_m E_A \exp(\mathrm{i}k_0 d)$, where $k_0 = \omega/c = 2\pi/\lambda_0$ is the incident beam's wave-number. Similarly, E_A is the superposition of $\tau_m E_0$, which passes through the first mirror, and a certain fraction of E_B, after E_B is phase-shifted (due to propagating a distance d through the gap) and bounced off the back-side of the front mirror. You must write down all the relevant relations, then eliminate E_A, E_B, and E_0, to arrive at expressions for ρ and τ as functions of ρ_m, τ_m, $k_0 = \omega/c$, and d.