Opti 501 **Final Exam** (12/12/2017) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) A plane-wave propagates in a medium described by the permeability $\mu(\omega) = 1.0$ and the refractive index $n(\omega) = \sqrt{\varepsilon(\omega)} = n'(\omega) + \mathrm{i}n''(\omega)$. The oscillation frequency is ω, and the propagation direction is the *real-valued* unit-vector $\hat{\boldsymbol{\kappa}} = \hat{\kappa}_x\hat{\boldsymbol{x}} + \hat{\kappa}_y\hat{\boldsymbol{y}} + \hat{\kappa}_z\hat{\boldsymbol{z}}$, so that the k-vector may be written as $\boldsymbol{k} = k\hat{\boldsymbol{\kappa}}$, with k being the (generally complex-valued) wave-number.

3 pts a) Use the dispersion relation to write an expression for the plane-wave's k-vector, clearly identifying the real and imaginary parts of the wave-number $k(\omega) = k'(\omega) + \mathrm{i}k''(\omega)$.

5 pts b) Write expressions for the field amplitudes $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ of the above plane-wave. You may assume that E_{x0} and E_{y0} are arbitrarily specified. However, E_{z0} is not free, and must be specified in terms of E_{x0}, E_{y0}, and the various components of $\hat{\boldsymbol{\kappa}}$. Similarly, the components of the magnetic field amplitude (H_{x0}, H_{y0}, H_{z0}) must be specified in terms of $\boldsymbol{E}_0$, $n(\omega)$, and $\hat{\boldsymbol{\kappa}}$.

5 pts c) Find the plane-wave's time-averaged Poynting vector $\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}(\boldsymbol{E} \times \boldsymbol{H}^*)$. (The vector identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ may be helpful here.) Examine the dependence of $\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle$ on the system parameters such as n', n'', $\hat{\boldsymbol{\kappa}}$, the E-field intensity $\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*$, the frequency ω, and the vacuum wavelength $\lambda_0 = 2\pi c/\omega$. Identify the absorption coefficient of the medium along the propagation direction $\hat{\boldsymbol{\kappa}}$.

Hint: If the time-averaged Poynting vector drops as $\exp(-\alpha d)$, where d is distance along the propagation direction, then α is said to be the absorption coefficient. (In contrast, n'' is often referred to as the extinction coefficient.)

Problem 2) In the Lorentz oscillator model, when the excitation frequency ω is near one of the material medium's resonance frequencies, say, ω_0, one is allowed to use a real-valued positive constant χ_b — the so-called "background susceptibility" — to represent the contributions of all the distant resonance frequencies to the overall susceptibility of the medium. The effective electric susceptibility in the vicinity of ω_0 could then be written as

$$\chi_e(\omega) = \chi_\mathrm{b} + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega}.$$

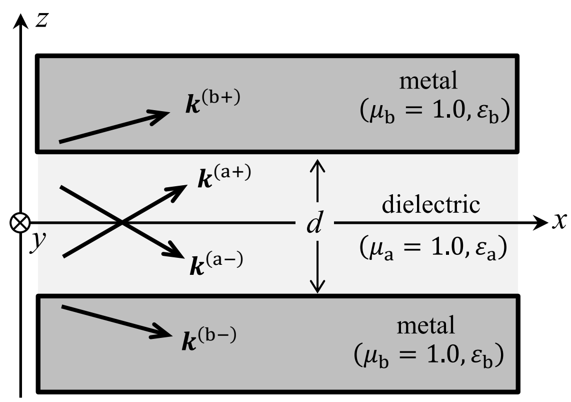
3 pts a) Write the dielectric function $\varepsilon(\omega) = \varepsilon'(\omega) + \mathrm{i}\varepsilon''(\omega)$ of the above material, and identify its real part $\varepsilon'(\omega)$ and imaginary part $\varepsilon''(\omega)$ as separate functions of the excitation frequency ω.

2 pts b) Assuming that the material permeability is $\mu(\omega) = 1.0$, write an expression for the (complex) refractive index $n(\omega)$ of the material medium in terms of $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$.

4 pts c) Let $f(x) = \sqrt{1 + x}$ be a function of a (generally complex-valued) variable x. Expand $f(x)$ in a Taylor series around the point $x = 0$, clearly identifying the 0^{th} order term, the 1^{st} order term, and the 2^{nd} order term of the Taylor series.

4 pts d) Assuming the imaginary part of $\varepsilon(\omega)$ is much smaller than its real part, i.e., $|\varepsilon''/\varepsilon'| \ll 1$, use the 0^{th} and 1^{st} order terms of the Taylor series expansion obtained in part (c) above to derive an approximate expression for the real and imaginary parts, $n'(\omega)$ and $n''(\omega)$, of the refractive index $n(\omega)$ obtained in part (b) above.

Problem 3) A transparent dielectric slab of thickness d and refractive index $n_\text{a} = \sqrt{\varepsilon_\text{a}}$ is sandwiched between two thick, identical metallic plates of (complex) dielectric constant ε_b. A guided wave of frequency ω, launched from the left-hand-side, is trapped within the dielectric slab while propagating forward along the x-axis. The symmetry of the problem allows the guided mode to be represented by a superposition of two plane-waves having wave-vectors $\boldsymbol{k}^{(\text{a}\pm)} = k_x\hat{\boldsymbol{x}} \pm k_z^{(\text{a})}\hat{\boldsymbol{z}}$, while the plane-waves within the upper and lower metallic plates have wave-vectors $\boldsymbol{k}^{(\text{b}\pm)} = k_x\hat{\boldsymbol{x}} \pm k_z^{(\text{b})}\hat{\boldsymbol{z}}$. In this problem, all the plane-waves are assumed to be linearly polarized along the y-axis (i.e., case of s-polarization).



3 pts a) Use the dispersion relation to write expressions for $k_z^{(\text{a})}$ and $k_z^{(\text{b})}$ in terms of ε_a, ε_b, ω, k_x, and the speed c of light in vacuum.

3 pts b) Denoting the corresponding E-field amplitudes by $E_0^{(\text{a}\pm)}\hat{\boldsymbol{y}}$ and $E_0^{(\text{b}\pm)}\hat{\boldsymbol{y}}$, write expressions for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the four plane-waves within their respective media.

3 pts c) Write the boundary conditions pertaining to the continuity of $\boldsymbol{E}_\parallel$ and $\boldsymbol{H}_\parallel$ at the metal-dielectric interfaces located at $z = \pm d/2$.

2 pts d) Find the allowed values of $E_0^{(\text{a}+)}/E_0^{(\text{a}-)}$ by comparing the H_x/E_y ratios at the two boundaries.

3 pts e) Using the H_x/E_y ratio at the upper (or lower) boundary in conjunction with each of the solutions obtained in part (d), find a unique equation for k_x in each case. (These equations, commonly known as the characteristic equations of the waveguide, must be solved numerically to yield all the allowed values of k_x at the given frequency ω. Here you are *not* being asked to *solve* the characteristic equations; your only task is to *derive* the equations.)