Opti 501 **Final Exam** (12/15/2015) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1) In the Lorentz oscillator model, the excitation field $\boldsymbol{E}(t) = E_{x0}\cos(\omega t)\,\hat{\boldsymbol{x}}$ produces a displacement of the electron $x(t) = |x_0|\cos(\omega t - \varphi_0)$ where

$$|x_0|\exp(\mathrm{i}\varphi_0) = \frac{(q/m)E_{x0}}{\omega^2 - \omega_0^2 + \mathrm{i}\gamma\omega}.$$

Here the electron's charge and mass are $-q$ and m, respectively, $\omega_0 = \sqrt{\alpha/m}$, where $\alpha \geq 0$ is the spring constant, and $\gamma = \beta/m$, where $\beta \geq 0$ is the friction coefficient. Now, according to Poynting's theorem, the time-rate-of-exchange of energy between the E-field and a localized electric dipole $\boldsymbol{p}(t)$ is

$$\frac{d\mathcal{E}(t)}{dt} = \boldsymbol{E}(t)\cdot\frac{d\boldsymbol{p}(t)}{dt}.$$

The energy $\mathcal{E}(t)$ appears in the form of kinetic energy $\mathcal{E}_K(t)$ and potential energy $\mathcal{E}_P(t)$ of the dipole, in addition to the lost energy $\mathcal{E}_L(t)$, which is dissipated via friction.

3 Pts a) Write an expression for the (real-valued) dipole moment $\boldsymbol{p}(t)$ of the Lorentz atom, then express $d\mathcal{E}(t)/dt$ as a function of ω, q, E_{x0}, $|x_0|$, and φ_0.

3 Pts b) Considering that $\mathcal{E}_K(t) = ½ m v_x^2(t)$, where $v_x(t) = dx(t)/dt$ is the electron's velocity, write an expression for the time-rate-of-change of the kinetic energy, $d\mathcal{E}_K/dt$, of the Lorentz atom.

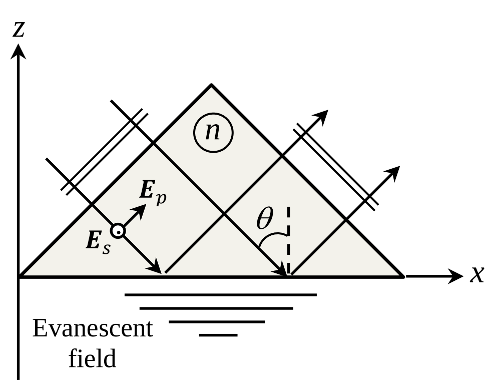
3 Pts c) Given that $\mathcal{E}_P(t) = ½\alpha x^2(t)$, write an expression for the time-rate-of-change of the potential energy, $d\mathcal{E}_P/dt$, of the Lorentz atom.

3 Pts d) The friction force acting on the electron is $F_x = -\beta dx(t)/dt$. When the electron moves a distance Δx, the work done (i.e., energy spent) against the friction force will be $-F_x\Delta x$. Write an expression for the time-rate-of-expenditure of energy against the friction force, $d\mathcal{E}_L/dt$, within the Lorentz atom.

3 Pts e) Show that your answer to part (a) is precisely equal to the sum of the three expressions that you found in parts (b), (c), and (d).

Hint: $\cos a\cos b - \sin a\sin b = \cos(a+b)$.

Problem 2) Incident at an angle θ at the bottom of a glass prism of refractive index $n(\omega)$ is a circularly-polarized plane-wave of frequency ω. The incidence angle is greater than the critical angle θ_c of total internal reflection. You may assume that the permeability $\mu(\omega)$ of the dielectric material (i.e., glass prism) is equal to 1.0. In the free-space region below the prism, the electromagnetic field is evanescent. (Note that all three components of both $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are present in this problem.)



3 Pts a) Write expressions for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the incident beam inside the glass prism. You may use a plus sign $(+)$ for one sense of circular polarization, and a minus sign $(-)$ for the opposite sense.

3 Pts b) Using the Fresnel reflection coefficients ρ_p and ρ_s, write expressions for the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the reflected beam inside the glass prism.

3 Pts c) Use the Fresnel transmission coefficients τ_p and τ_s to write an expression for the evanescent $\boldsymbol{E}$ field that resides in the free-space region below the prism. (**Hint**: You may use Maxwell's first equation, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, to express E_z^{t} in terms of E_x^{t}.)

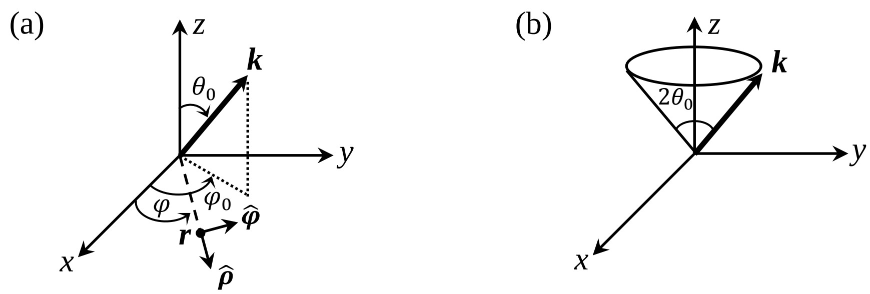
2 Pts d) Use Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, to determine the evanescent $\boldsymbol{H}$ field in terms of the evanescent $\boldsymbol{E}$ field components that you obtained in part (c).

1 Pt e) Calculate $\langle S_x\rangle$, the time-averaged x-component of the Poynting vector of the evanescent wave, and confirm that it has a non-vanishing value. Verify that $\langle S_x\rangle$ does not change if the sense of polarization of the incident beam is switched between right- and left-circular.

1 Pt f) Calculate $\langle S_y\rangle$, the time-averaged y-component of the Poynting vector of the evanescent wave, and confirm that it does not vanish. Moreover, show that $\langle S_y\rangle$ switches sign when the sense of incident polarization changes from right- to left-circular.

1 Pt g) Calculate $\langle S_z\rangle$, the time-averaged z-component of the Poynting vector of the evanescent wave, and confirm that it is precisely equal to zero.

Problem 3) Consider a homogeneous plane-wave of frequency ω, propagating in free space along the direction of a k-vector specified by its polar and azimuthal angles (θ_0,φ_0), as shown in Fig.(a). The plane-wave is p-polarized relative to the plane of its k-vector and the z-axis.



4 Pts a) In the cylindrical coordinate system (ρ,φ,z), write the components k_z and k_ρ of the k-vector in terms of ω, c, and θ_0.

4 Pts b) Write the $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\varphi}}$, and $\hat{\boldsymbol{z}}$ components of the plane-wave's $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as functions of E_0, Z_0, θ_0, φ_0, k_z, k_ρ, ω, ρ, φ, z, and t.

3 Pts c) Let there be a superposition of a continuum of plane-waves whose k-vectors form a right-circular cone with an apex angle of $2\theta_0$, as shown in Fig.(b). All plane-waves are p-polarized, have the same amplitude and phase, and their k-vactors have a fixed θ_0 but a variable φ_0 in the range of $[0,2\pi]$. Integrate over φ_0 the fields obtained in part (b) in order to arrive at the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the superposed plane-waves at all points in the cylindrical coordinate system.

Hint: $\int_0^{2\pi}\exp(\mathrm{i}x\cos\varphi)\,d\varphi=2\pi J_0(x)$ (G&R 3.915-2); $J_0'(x)=-J_1(x)$ (G&R 8.473-4).
